Numerical methods for Hamilton Jacobi Bellman (HJB) equations and Mean Field games (MFG)

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MFG workshop : Economists meet mathematicians

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Introduction – Motivation

- Nowadays, heterogeneous agents models are ubiquitous is economic theory
  - Started in the 1990s with Bewley-Huggett-Aiyagari framework for household inequality and Hopenhayn for entry and exit of firms
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  - When formalized in continuous time, these models take the form of systems of PDE, called MFG system
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- What are we talking about ?
  - ▷ A Hamilton-Jacobi-Bellman : backward in time
    How the agent value/decisions change when distribution is given
  - ▷ A Kolmogorov-Forward (Fokker-Planck) : forward in time
    How the distribution changes, when agents control is given
  - ▷ These two relations are *coupled* :
    e.g. due to equilibrium prices (r_t/w_t) ⇒ *need to look for a fixed point*

Introduction – recent progress and open questions

- Recent progress :
  - Very fast to compute the stationary equilibrium
  - Recent methods that rely on linearity of the model (more on this later)
- However, plenty of open questions
  - No ideal methods with transition path.
  - Simulation with aggregate shocks/common noise still impossible
  - Understanding the gain and losses induced by simplification :
    - See other set of slides about methods with aggregate shocks
- Today : numerical methods for
  1. "Standard" HJB, extensions to MFG $\Rightarrow$ ex. w/ Aiyagari (1994)
  2. Impulse control and HJB-VI and MFG $\Rightarrow$ ex. w/ Hopenhayn (1992)
  3. Introduction to common noise : MIT shocks and Jacobian methods

Baseline model – Aiyagari model

- Let us recap the Aiyagari framework :
  - Will use it thoroughly as an example for the different algorithms
  - Continuous time version of the stationary case :

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  - Continuous time version of the stationary case :
  - Household :
    - States variables : wealth a and labor prod. z ; control : consumption c
    - Idiosyncratic fluctuations in z (Pure jump/Jump-drift process)
    - State constraint (no borrowing) $a \geq \underline{a}$
    - Maximization :

$$\max_{\{c_t\}} \mathbb{E}_{t_0} \int_{t_0}^{\infty} e^{-\rho t} u(c_t) dt \qquad da_t = \underbrace{(z_t w_t + r_t\, a_t - c_t)}_{= s^\star(t,a,z)} dt \qquad a|_{t_0} = a_0$$

  - Neoclassical firms : $Y_t = Z_t K_t^{\alpha} z_{av}^{1-\alpha}$
    - Interest rate : $r_t = \alpha\, Z_t K_t^{\alpha-1} z_{av}^{1-\alpha} - \delta$ & wage $w_t = (1-\alpha)\, Z_t K^{\alpha} z_{av}^{-\alpha}$
    - Capital demand $K_t(r) := \left(\frac{\alpha Z_t}{r_t + \delta}\right)^{\frac{1}{1-\alpha}} z_{av}$
  - Discrete time version here

MFG system - Aiyagari model - 1

- Original Aiyagari model :
  - Idiosyncratic noise on z_j is a Markov jump-process, $1 \leq j \leq n_z$, intensity λ_j and $z \overset{\mathcal{L}}{\sim} \phi(\cdot)$ conditional on jumping
  - $da_t = (\underbrace{z_t w_t + r_t\, a_t - c_t}_{=s(t,a,z,r_t(g),c_t)=s^{\star}(t,a,z)})dt$ and state space : $(a,z_j) \in [\underline{a},\infty) \times \{z_1,\ldots,z_{n_z}\} =: \mathbb{X}$

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$$-\partial_t v(t,a,z_j) + \rho v(t,a,z_j) = \max_c u(c) + \partial_a v(t,a,z_j)\, s(t,a,z_j)$$

$$\forall\ (t,a,z)\in[0,T)\times\mathbb{X} \qquad + \lambda_j \sum_{-j} \phi(z_{-j})\big(v(t,a,z_{-j}) - v(t,a,z_j)\big)$$

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$$S_t(r_t) := \sum_{z_j} \int_{\underline{a}}^{\infty} a\, g(t,da,z_j) = K_t(r_t)$$

$$v(T,a,z) = v_\infty(a,z) \qquad g(t_0,a,z) = g_0(a,z) \qquad \forall\ (a,z_j) \in \mathbb{X}$$

MFG system - Aiyagari model - 2

- Diffusion-version of Aiyagari model :
  - Idiosyncratic noise z is now a diffusion process $dz = \mu(z)dt + \sigma^2 dB_t$.
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The algorithm : an overview

- Aim : find the stationary equilibria : i.e. the functions v and g, over $[0, T]$ and the interest rate path r.
- General structure :
  1. Guess interest rate path r^ℓ, compute capital demand $K(r^\ell)$ & wages $w(K)$
  2. Solve the HJB using finite differences (semi-implicit method) : obtain $s^\ell{}_{(a,z_j)}$ and then $v^\ell{}_{(a,z_j)}$, by a system of sort : $\rho\,\mathbf{v} = \mathbf{u}(\mathbf{v}) + \mathbf{A}(\mathbf{v};r)\mathbf{v}$
  3. Using $\mathbf{A}^T$, solve the FP equation (finite diff. system : $\mathbf{A}(\mathbf{v};r)^T\mathbf{g} = 0$), and obtain $g_{(a,z_j)}$
  4. Compute the capital supply $S(\mathbf{g}, r) = \sum_j \int_a^\infty a\, g_{(a,z_j)} da$
  5. If $S(r) > K(r)$, decrease $r^{\ell+1}$ (update using bisection method), and conversely, and come back to step 2.
  6. Stop if $S(r) \approx K(r)$

The algorithm, advantages relative to discrete time :

1. Borrowing constraint only appears in the boundary conditions
   - FOCs $u'(c(a)) = \partial_a v^j(a)$ and HJB eq. always holds with equality
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4. HJB and FP are coupled
   - The matrix to solve FP is the transpose of the one of HJB.
   - Why ? Operator in FP is simply the 'adjoint' of the operator in HJB : 'Two birds one stone'
   - Specificity of MFG !

The algorithm : transition dynamics

- The algorithm for transition dynamics :
  - Discretization : $v_{i,j}^n$ and $g_{i,j}^n$ stacked into v^n and g^n
  - Somehow, it is more specific to Mean Field Games :

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- Take advantage of the backward-forward structure of the MFG
  - Make a guess r^ℓ_t $(t = 1, \dots, N)$ on the *path* interest rates.
  - Solve the HJB (implicit scheme), given terminal condition ;

$$-\frac{v^{n+1} - v^n}{\Delta t} + \rho v^{n+1} = u^n + \mathbf{A}(v^{n+1}; r^n)\, v^{n+1}$$

$$v^N = v_\infty \qquad \text{(terminal condition = steady state)}$$

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  - Solve the FP forward, given the initial condition

$$\frac{g^{n+1} - g^n}{\Delta t} = \mathbf{A}(v^n; r^n)^T g^{n+1}$$

$$g^1 = g_0 \qquad \text{(initial condition)}$$

  - Update the interest rates path

The algorithm for HJB : Finite difference

- Finite difference scheme :
  - Discretize the state-space a_i for $i = 1, \dots n_a$ and z_j for $j = 1, \dots, n_z$, and time $t = 1, \dots, N$

$$\partial_a v(a_i, z_j) \approx \frac{v_{i+1,j} - v_{i,j}}{\Delta a} \equiv v'_{i,j,F} \qquad \partial_a v(a_i, z_j) \approx \frac{v_{i-1,j} - v_{i,j}}{\Delta a} \equiv v'_{i,j,B}$$

$$\partial^2_{zz} v(a_i, z_j) \approx \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{\Delta a} \qquad \partial_t v(t, a_i, z_j) \approx \frac{v^{n+1}_{i,j} - v^n_{i,j}}{\Delta t}$$

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- Looking for v^{n+1} as a function v^n. Implicit method
  - Inversion of time : HJB runs backward – from $T \equiv 1$ to $t_0 \equiv N$

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- Main issue for HJB : controls depend on v^{n+1} through the max :
  - We rely on semi-implicit methods for control c : use $c^n_{i,j}$ instead of $c^{n+1}_{i,j}$ for the fully-implicit

$$c^n_{i,j} = (u')^{-1}(v^{n\,\prime}_{i,j}) \qquad \text{instead of} \qquad v^{n+1\,\prime}_{i,j}$$

The algorithm for HJB : Finite difference

- Optimality condition in the maximization of the HJB
  - Hamiltonian and FOC : $p \equiv \partial_a v$

$$H(g,p) := \max_c u(c) + s(r(g),c)p \qquad \Rightarrow \qquad u'(c) = p$$

$$c^\star = (u')^{-1}(p) \qquad\qquad s^\star(t,a,z) = \partial_p H(g, \partial_a v_{(t,a,z)})$$

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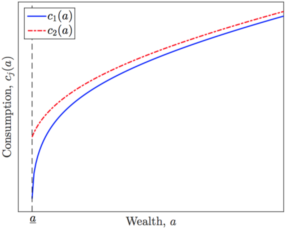
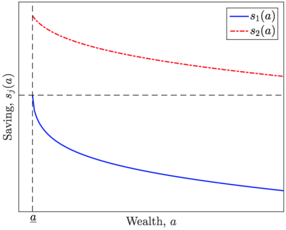
- Upwind scheme :
  - Choose direction of difference as fct of the sign of drift :

- If $s^\star(t,a,z) > 0$ choose $p \equiv \partial_a v = v'_{i,j,F}$
- If $s^\star(t,a,z) < 0$ choose $p \equiv \partial_a v = v'_{i,j,B}$





The algorithm for HJB : Finite difference scheme

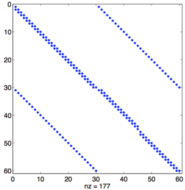
- Boundary conditions :
  - State constraint in a

$$a \geq \underline{a} \qquad s(t,\underline{a},z) \geq 0 \qquad \Rightarrow \qquad \underline{c}_t = r_t \underline{a} + w_t z$$

$$v'_{1,j,B} = \partial_a v(t,\underline{a},z) = u'(\underline{c}_t)$$

  - Boundaries in z : implied by the reflecting barrier

$$\partial_z v(t,a,\underline{z}) = \partial_z v(t,a,\bar{z}) = 0 \qquad \Rightarrow \qquad v'_{i,j,F} = v'_{i,j,B} = 0$$



- All this determines the operator in HJB
  - The matrix $\mathbf{A}(v^n; r^n)$ is sparse
  - Fast to invert
- Solving the (now) linear system for the HJB

$$-\frac{v^{n+1} - v^n}{\Delta t} + \rho v^{n+1} = u^n + \mathbf{A}(v^n; r^n)\, v^{n+1}$$

The algorithm for HJB : theoretical result

- Barles and Souganidis (1991) : *Convergence of approximation schemes for fully nonlinear second order equations*, Asymptotic Anal. 4
  - Generalization of Souganidis (1985) *Approximation Schemes for Viscosity Solutions of Hamilton-Jacobi Equations*, J. Differential Equations
  - Result much more general than most econ application
- This numerical solution v^ρ converges uniformly ($\rho \to 0$) to the unique (viscosity) solution v of the HJB, under some conditions :
  1. Monotonicity ($\mathbf{A}w \geq \mathbf{A}u$ if $w \leq u$)
  2. Consistent ($\limsup_{\xi \to 0, \rho \to 0} \mathbf{A}(v + \xi) = \mathcal{A}(v)$)
  3. Stability (v^ρ is bounded uniformly in ρ)
- The matrix is monotonous :
  - The upwind scheme insures the convergence of the algorithm

The algorithm : generalization to MFGs

- Fokker Planck solved immediately

$$\frac{g^{n+1} - g^n}{\Delta t} = \mathbf{A}(v^n; r^n)^T g^{n+1}$$

  - The finite difference scheme is analogous, except that the upwind scheme is reversed
  - Additional gain : Property that the operator in FP $\mathcal{A}^* / \mathbf{A}(v^n; r^n)^T$ is the adjoint of $\mathcal{A} / \mathbf{A}(v^n; r^n)$ in HJB

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- Mean Field Games system : (v_t, g_t, r_t)
  - Idea analogous to Schauder fixed point used in the proofs :
    1. Start from a guess $\{g_t\}_{t\in[0,T]}$ and $\{r_t(g_t)\}_t$
    2. Solve for $\{v_t\}_t$ in the HJB
    3. Solve for $\{\widetilde{g}_t\}_t$ in the FP
    4. If $||\widetilde{g}_t - g_t||_\infty > \varepsilon$, update

The algorithm : generalization to MFGs

- Finding an equilibrium path $\{r_t\}_t$
  - Such that HJB, FP and market clearing $S(t, r_t) := \int_{\mathbb{X}} ag(t,a,z) = K(r_t) =: K_t$ hold
- Question : how do you update ?
  - No systematic answer !
  - In practice, update path of r_t or K_t, for example, at step ℓ :

$$r_t^{\ell+1} = r_t^{\ell} + \theta_\ell e^{-\alpha t} \widehat{S}_t$$

with $\widehat{S}_t := K_t - S_t$ or $\widehat{S}_t = \partial_t(K_t - S_t)$

The algorithm : generalization to MFGs

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- Performance of the algorithm :
  - Very fast : Stationary equilibrium in less than 0.3 sec
  - Fast for partial-equilibrium solution $\{v_t, g_t\}$ given path r_t
  - But may be super slow for finding a fixed point (i.e. in r_t)

Convergence theorems

- Achdou and Capuzzo-Dolcetta (2010) *Mean field games : numerical methods*. SIAM J. Numer. Anal., 48(3) :1136-1162
  - Stationary case
- Achdou and Porretta (2016) *Convergence of a finite difference scheme to weak solutions of the system of partial differential equations arising in mean field games*. SIAM Journal on Numerical Analysis, 54(1), 161-186.
  - Time varying case, implicit in time $n = 1, \ldots N$
- Results for typical second order MFG
  - Diffusion : $\nu \Delta v$, Separable Hamiltonian $H(x,m,p) = \widetilde{H}(x,p) - f(m)$, periodic sets $\mathbb{T}$
- Plenty of other numerical methods for variational MFG :
  - MFG system as optimality condition of a control problem
  - Can solve directly the optimal planning/optimal transport problem
  - Rely on calculus of variation and convex duality

Convergence theorem

- Finite difference scheme : h, $v \approx V^{h,\Delta t}$ and $g \equiv M^{h,\Delta t}$

$$(D_t V)^n - \nu\left(\Delta_h V^n\right)_{i,j} + g(x_{i,j}, [D_h V^n]_{i,j}) = V_h[M^{n+1}]$$
$$(D_t M)^n - \nu\left(\Delta_h M\right)_{i,j} + \mathcal{B}_{i,j}(V, M) = 0$$
$$V_i^N = \phi(M_i^N) \qquad\qquad M_i^0 = m_0(x_i)$$

  - Assumption on Hamiltonian $g(\cdot)$:
    - Monotonicity : $\forall x, g(x, [p^+, p^-])$ non decreasing in p^+ and non increasing in p^-
    - Consistency : $\forall x, g(x, [p, p]) = H(x, p)$
    - Differentiability H is a class $\mathcal{C}^1$
    - Convexity of $g(x, [p^+, p^-])$ in p
    - Growth condition :
      $g_q(x, q) \cdot q - g(x, q) \geq c_1 |g_q(x, q)|^2 - c_2 \qquad |g_q(x, q)| \leq c_3 |q| + c_4$

- Theorem 3 : Convergence in norm L^p to the solution : $V^{h,\Delta t} \to v$ and $M^{h,\Delta t} \to v$, where (v, g) is a weak solution of the MFG system

Introduction to Impulse control and MFG

- Many economic problem feature impulse control (with Mean Field interaction !) :
  - Not considered as such since all the variables ”jump” in discrete time
  - Fixed cost / non-convex cost of controls / Stopping time problem
  - Create “inaction regions” when agents don’t exert control and thresholds where they pay the cost and jump

Introduction to Impulse control and MFG

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  - Create “inaction regions” when agents don’t exert control and thresholds where they pay the cost and jump
- Applications to :
  - Entry and Exit of Firms : Hopenhayn (1992), application to international trade : Melitz (2003)
  - Pricing models à la Golosov and Lucas Jr (2007) and Calvo+ (c.f. Alvarez and Lippi (2014), Alvarez et al. (2016))
  - Heterogeneous firms with lumpy invest^{nt}, e.g. Khan and Thomas (2008), Winberry (2016a)
  - ▷ Book : “Economics of Inaction : Stochastic Control models with fixed costs” Stokey (2009)
- I’ll cover the framework of Hopenhayn since it’s the most simple and can be easily generalized.

Baseline model – Hopenhayn model

- Let us introduce a simplified version of Hopenhayn :
  - Time varying case (while stationary in the original article)
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  - Firm :
    - States variables : production z ; control : employment n and exit time τ
    - Idiosyncratic fluctuations in z (Jump-drift process / Diffusion : Brownian with reflecting barriers $[\underline{z}, \bar{z}]$
    - Maximization :

$$v(z_{t_0}) = \max_{\{n_t\}} \mathbb{E}_{t_0} \int_{t_0}^{\tau} e^{-\rho t} \pi(z_t, n_t) dt + e^{-\rho\tau} v^{\star}$$

$$\pi(z_t, n_t) = p_t f(z_t, n_t) - w_t n_t - c^f \qquad dz = \mu(z)dt + \sigma^2 dB_t \qquad z|_{t_0} = z_0$$

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  - Mean field interaction through price p and wage w
    - Wage : $w = W(N)$ where $n^\star(z)$ optimal employment and aggregate employment $N = \int_{\underline{z}}^{\overline{z}} n^\star(z) g(z) dz$
    - Price of good : $p = D(Q)$ where $q^\star(z) := f(z, n(z))$ and aggregate good supply $Q = \int_{\underline{z}}^{\overline{z}} q^\star(z) g(z) dz$
  - Plenty of extension : endogenous entry, multiple state variables

MFG system - Hopenhayn model - 1

- Hopenhaym model :
  - Profit $\pi(z_t, n_t) = p_t f(z_t, n_t) - w_t n_t - c_f$ and Coupling $p_t = \mathcal{P}(g_t)$ and $w = \mathcal{W}(g)$
  - Inaction region : $\mathcal{Z} \subset [\underline{z}, \bar{z}]$

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$$v(T,z) = v_\infty(z) \qquad g(t_0,z) = g_0(z) \qquad \forall\ z\in[\underline{z},\bar{z}]$$

MFG system - Hopenhayn model - 2, HJB-VI

- Hopenhayn model, reformulation with variational inequality :
  - Optimal choice of labor $n(z) = \left(p\partial_n f(z,\cdot)\right)^{-1}(w)$ & $\pi^\star(z) = \pi(z, n(z))$
  - Operator : $(\mathcal{A}_t v)(t,z) = \partial_z v(t,z)\,\mu(t,z) + \frac{\sigma^2(z)}{2}\partial^2_{zz} v(t,z)$

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  - Exit : Integrating the FP, we obtain mass of firm exit : $m_t = -\int_{\mathcal{Z}} \mathcal{A}^* g(t,x)dx$ over Inaction region $\mathcal{Z}$
  - Adjoint : $(\mathcal{A}_t^* g)(t,z) = -\partial_z\left[g(t,z)\,\mu(t,z)\right] + \partial^2_{zz}\left[\frac{\sigma^2(z)}{2} g(t,z)\right]$

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  - Reformulate as a Variational inequality :

$$\forall\ (t,z)\in[0,T)\times[\underline{z},\bar{z}] \qquad \min\left\{-\partial_t v(t,z) + \rho v(t,z_j) - \pi^\star(z) - \mathcal{A}v(t,z); v(z) - v^\star\right\} = 0$$

$$\forall (t,z)\in[0,T)\times\mathcal{Z} \qquad \partial_t g(t,z) = \mathcal{A}^* g(t,z) + m_t \psi(z)$$

$$\forall\ z\in[\underline{z},\bar{z}] \qquad v(T,z) = v_\infty(z) \qquad g(t_0,z) = g_0(z)$$

Numerical methods for HJB-VI

- Solving the QVI-HJB with Implicit scheme finite difference methods :
  - Splitting the problem :
  - In the inaction region $\mathcal{Z}$ the problem is the same as above and we obtain : $$-\frac{v^{n+1} - v^n}{\Delta t} + \rho v^{n+1} = \pi^n + \mathbf{A}(v^n; p^n, w^n)\, v^{n+1}$$
  - Action $v^{n+1} = v^*$
- Can be reformulated as a Linear Complementarity problem (LCP) of the form :

$$\begin{aligned} (v - v^*)^T (\overbrace{[\rho - \tfrac{1}{\Delta t} - \mathbf{A}]}^{=B}\, v - \pi + \tfrac{v^o}{\Delta t}) &= 0 \\ v - v^* &\geq 0 \\ B\, v - \pi + \tfrac{v^o}{\Delta t} &\geq 0 \end{aligned}$$

- Some solvers exists to handle this LCP problems
- Other iterative methods exist like PSOR (Projected Successive Over Relaxation) or semi smooth Newton Methods

Adding aggregate uncertainty – Common noise

Adding aggregate uncertainty – Common noise

- What are the problems with aggregate risk ?
  - Aggregate shocks will affects the shape of the distribution
  - Agents needs to forecast its motion (of $g_t(\cdot)$) to make expectations about future prices ($r_t \dots$) and value v_t

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    - Only in case of strategic complementarity – coupling of HJB with FP.
  - The distribution $g(t, a, z_j)$, which is an infinite-dimensional object, becomes a state variable for each agent.
  - This changes for each path/history of aggregate shocks Z_t

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  - This changes for each path/history of aggregate shocks Z_t
- Examples :
  - AR(1)-change in agg. TFP Z_t : $dZ_t = \theta(\bar{Z} - Z_t)dt + \sigma dB_t$
  - Could also consider :
    - Shock to credit constraint $\underline{a}$ or to asset supply (gov^{nt} bond issuance)
    - Demand shocks/patience shock ρ
    - Change in idiosyncratic volatility $\sigma_z \equiv \mathbb{V}\text{ar}(z)$ or transition probas λ

MIT shocks : unexpected shocks

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  - Z_t is subject to a one-time shock on dB_t, i.e. normal $\mathcal{N}(0, \sigma)$
  - Then Z_t follows the OU-(AR(1)) drift process $dZ_t = \theta(\bar{Z} - Z_t)dt$

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- *Main idea :*
  - Agents do no anticipate this and hence do not draw expectations
    - v_0 does not include the potentiality of such shocks
    - Once the shock is "revealed" there is no more uncertainty on the path of Z_t

  $\Rightarrow$ Certainty equivalence **(CE)** :
    - No influence of variance σ : only size of the shock matters
    - CE typically holds in Linear-Quadratic model with (additive) shocks : quadratic utility/objective fct. and linear transition/policy functions
    - (good approximation for more general models ?)

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- Solution method :
  - ▷ Almost no difference compared to deterministic case (cf. above)

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  - Almost no difference compared to deterministic case (cf. above)
  1. Solve the HJB using backward induction : start from steady state v_T where T large (close to stationary)
  2. Solve the KF forward : start from the “before-shock” steady state g_0
  3. Find the equilibrium fixed-point, by iterating on the entire *path* of prices $\{r_t\}_{t\in[0,T]}$
- Method most commonly used as a starting point
  - Certainty equivalence and no anticipation
  - Often implies small GE effects (little price effects)

Combining Linearization and MIT shocks : BKM

- Boppart, Krusell and Mitman (2018)
  - *Exploiting MIT shocks in heterogeneous-agent economies : the impulse response as a numerical derivative*, JEDC
  - Recent generalization by Auclert et al. (2019) and recent work by Kaplan-Moll-Violante
- *Main idea :*
  - Combining non-linearity of responses to MIT shocks
  - With linearity assumption to combine multiple shocks
  - IRF of an MIT shock is a derivative of the system :

    $\Rightarrow$ we ”just” need to “compute” it once !

Combining Linearization and MIT shocks : BKM

- More details on BKM
  - Sequential representation of heterogeneous agents models :
  - Express aggregate variables K_t (or C_t) as a fct of past shocks on Z_t
    - Sequence form :

$$dK_t = \mathcal{K}(\{dZ_s\}_{s\leq t}) \approx \mathcal{K}(dZ_t, dZ_{t-1}, \dots)$$

    - vs. Recursive form : $K_t = \widetilde{\mathcal{K}}(\Theta_t)$ with Θ_t states var. (v_t, g_t, r_t)
- Linearity assumption of the system :

$$\begin{aligned} dK_t &= \int_0^t \partial_{dZ_s}\mathcal{K}(0) dZ_s \\ &\approx \underbrace{\mathcal{K}(\varepsilon, 0, 0, \dots)}_{\substack{IRF \text{to a 1-time} \\ \varepsilon-\text{sized MIT shock} \\ \equiv \mathcal{K}_{dZ}(0)}} dZ_t + \mathcal{K}(0, \varepsilon, 0, \dots) dZ_{t-1} + \dots \end{aligned}$$

Combining Linearization and MIT shocks : BKM

- Solution method in practice :
  1. Simulate the IRF to a small (sized ε) MIT shocks :
     - Shock at date s gives IRF : $dK_t^s = \mathcal{K}(0, \ldots, \varepsilon, 0, \ldots)$
     - Such path represent the non-linear derivative $\partial_{dZ_s}\mathcal{K}(0)$ of the system to a shock
  2. Simulate a sequence of shocks $(\{dZ_s\}_{s \leq t}$
  3. Sum the IRF for different shock, rescaling by the size of the shock :

$$dK_t = \int_0^t \partial_{dZ_s}\mathcal{K}(0) dZ_s \approx \sum_s^t \frac{1}{\varepsilon} dK_t^s dZ_s$$

     - Possibility of testing the linearity assumption by changing the size/sign of ε

Linearization & MIT shocks – Extensions : SHADE

- Auclert, Bardóczy, Rognlie and Straub (2019)'s SHADE :
  - Equilibrium relations as the system :

$$H(K_t, Z_t) = 0$$

  - Linearizing :

$$H_K(\overline{K}, \overline{Z})dK_t + H_Z(\overline{K}, \overline{Z})dZ_t = 0$$

  - Path of capital as function of past shocks :

$$dK_t = \underbrace{-[\overline{H}_K]^{-1}\overline{H}_Z}_{\equiv \mathcal{K}_{dZ}(0)} dZ_t$$

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- $\overline{H}_K$ and $\overline{H}_Z$ called "sequence space Jacobians"
  - Need to be computed once
  - Sufficient statistics : all we need, to know the agg. system response
  - Fast : used in estimation (of shock process dZ_s)

Linearization & MIT shocks – Extensions : SHADE

- These "sequence space Jacobians" :
  - Are the sufficient statistics :
    - $\overline{H}_K, \overline{H}_Z$ and $\mathcal{K}_{dZ} \equiv -[\overline{H}_K]^{-1}\overline{H}_Z$ as a $T \times T$ matrix
    - IRF for a path $\{dZ_t\}_t$: $\approx$ derivative of system in response to shocks
    - "News" of different horizons s shocks : s-th columns of $\mathcal{K}_{dZ}$
    - Include "under the hood" the underlying heterogeneity

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    - Direct methods (finite difference)
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- Substantial speed gains :
  - Linearization and no need to recompute the Jacobian
  - Lots of clever methods :
    - Directed acyclic graph to exploit the sparsity of system : dimension reduction by composition of Jacobians along the blocks of this DAG
    - Likelihood-based estimation : feasible now for even large models

Conclusion

- Challenging problem and many different methods
- Stationary equilibria well understood
- No perfect solution for common noise – unfortunately
  - Every algorithm with its own way of bypassing difficulties
  - e.g. trade-off : Linearity/simplification for "speed"
    vs. Role for uncertainty/shape of distribution for "accuracy"
- Still lack of theoretical results on the strength of various methods
  - Global methods vs. Local perturbation/MIT shocks
  - Could compare them for various (closed-form) models

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- THANK YOU FOR YOUR ATTENTION !

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Aiyagari model without aggregate risk – discrete time

- Household :
  - Two states : wealth a and labor prod. z ; control consumption : c
  - Idiosyncratic fluctuation in z (Markov chain/AR(1) process)
  - State constraint (no borrowing) $a_t \geq \underline{a}$
  - Maximization :

$$\max_{c_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad c_t + a_{t+1} = z_t w_t + r_t\,(1 + a_t)$$

- Neoclassical firms : $Y_t = Z_t K_t^{\alpha} z_{av}^{1-\alpha}$
  - Interest rate : $r_t = \alpha\, Z_t K_t^{\alpha-1} z_{av}^{1-\alpha} - \delta$ & wage $w_t = (1-\alpha) Z_t K^{\alpha} z_{av}^{-\alpha}$
  - Capital demand $K_t(r) := \left(\frac{\alpha Z_t}{r_t + \delta}\right)^{\frac{1}{1-\alpha}} z_{av}$

Aiyagari model without aggregate risk – discrete time

- Equilibrium (recursive) relations :

Back

Aiyagari model without aggregate risk – discrete time

- Equilibrium (recursive) relations :
  - A Bellman equation : backward in time
    How the agent value/decisions change when distribution is given

$$v_t(a,z) = \max_{c,a'} u(c) + \beta \mathbb{E}\left[v_{t+1}(a',z') \middle| \sigma(z)\right]$$

$$s.t. \qquad c+a'=zw_t+r_t\,(1+a) \quad a' \geq \underline{a} \quad \Rightarrow \quad a'^{\star} = \mathscr{A}(a,z)$$

Back

Aiyagari model without aggregate risk – discrete time

- Equilibrium (recursive) relations :
  - A Bellman equation : backward in time
    How the agent value/decisions change when distribution is given
  - A Law of Motion of the distribution : forward in time
    How the distribution changes, when agents control is given

$$v_t(a,z) = \max_{c,a'} u(c) + \beta \mathbb{E}\left[v_{t+1}(a',z') \middle| \sigma(z)\right]$$

$$s.t. \qquad c+a'=zw_t+r_t\,(1+a) \quad a' \geq \underline{a} \quad \Rightarrow \quad a'^{\star} = \mathscr{A}(a,z)$$

$$\forall\ \widetilde{A} \subset [\underline{a},\infty) \qquad g_{t+1}(\widetilde{A},z') = \sum_z \pi_{z'|z} \int \mathbb{1}_{\{\mathscr{A}(a,z)\in\widetilde{A}\}} g_t(da,z)$$

Back

Aiyagari model without aggregate risk – discrete time

- Equilibrium (recursive) relations :
  - A Bellman equation : backward in time
    How the agent value/decisions change when distribution is given
  - A Law of Motion of the distribution : forward in time
    How the distribution changes, when agents control is given
  - These two relations are *coupled* :
    Through firm pricing (r_t & w_t) $\Rightarrow$ need to look for an eq. fixed point

$$v_t(a,z) = \max_{c,a'} u(c) + \beta \mathbb{E}\left[v_{t+1}(a',z') \middle| \sigma(z) \right]$$

$$s.t. \quad c+a'=zw_t+r_t\,(1+a) \quad a' \geq \underline{a} \quad \Rightarrow \quad a'^{\star} = \mathscr{A}(a,z)$$

$$\forall\ \widetilde{A} \subset [\underline{a},\infty) \qquad g_{t+1}(\widetilde{A},z') = \sum_z \pi_{z'|z} \int \mathbb{1}_{\{\mathscr{A}(a,z)\in\widetilde{A}\}} g_t(da,z)$$

$$S_t(r) := \sum_z \int_{\underline{a}}^{\infty} a\, g_t(da,z_j) = K_t(r)$$

Back