MICRO AND MACRO UNCERTAINTY

ANDREAS SCHAAB - JOB MARKET PAPER

Léo Aparisi, Thomas Bourany, Zhiyu Fu
THE UNIVERSITY OF CHICAGO

January 2020

Macro reading group

Introduction

- How does uncertainty interacts at the macro and the micro level?
  - Macro uncertainty: discount rate (demand) shocks, ZLB (non linearities).
  - Micro uncertainty: unemployment
- Endogenous link between micro-macro uncertainty:
  - Aggregate demand.
  - Job fluctuation.
- Develop a new global solution method to heterogeneous agents model with aggregate shocks
  - Reduce the dimensionality of the distribution by using a projection method.
- Add aggregate demand shocks and ZLB to HANK model of Kaplan, Moll, and Violante (2018).
  - Estimate that the welfare cost of business cycles increases to 3.9% of average consumption.

- 2 periods $t \in \{0, 1\}$
- Aggregate state of the economy at $t = 1$: $Y_1(\sigma\epsilon)$
  - $\epsilon \sim \mathcal{N}(0, 1)$ at $t = 0$
  - σ summarizes uncertainty
- Household i faces idiosyncratic risk on its employment status $z_{i,t} \in \{0, 1\}$
  - State-dependent transition probability $p_i(Y_1(\sigma\epsilon)) = \mathbb{P}(z_{i,1} = 1 | z_{i,0}, \epsilon)$
- Maximizes time 0 expected utility: $U(c_{i,0}) + \beta\mathbb{E}_0[c_{i,1}]$
- Savings decision with wage $\propto Y_t$, inelastic labor supply, and *fixed* return R:

$$\begin{aligned} c_{i,0} + a_{i,1} &= a_{i,0} + \gamma_i Y_0 z_{i,0} \\ c_{i,1} &= R a_{i,1} + \gamma_i Y_1 z_{i,1} \\ U'(c_{i,0}) &= \beta R \mathbb{E}_0\left[U'\left(c^u_{i,1}\right)\{1 - p_i(Y_1(\sigma\epsilon))\} + U'\left(c^e_{i,1}\right) p_i(Y_1(\sigma\epsilon))\right] \end{aligned}$$

- 2^{nd} order expansion around $\sigma = 0$:

$$c_{i,0}(\sigma) \approx c_{i,0}(0) + \frac{1}{2} \left. \frac{d^2 c_{i,0}}{d\sigma^2} \right|_{\sigma=0} \sigma^2$$

  - With:

$$\begin{aligned}
\frac{\left.\frac{d^2 c_{i,0}}{d\sigma^2}\right|_{\sigma=0}}{\beta R \mathrm{MPS}_{i,0}} =& \frac{U'''\left(c^e_{i,1}\right)}{U'''\left(c^0_{i,}\right)} \gamma_i^2 p_i \left(\frac{\partial Y_1}{\partial \sigma}\right)^2 + 2 \frac{U''\left(c^e_{i,1}\right)}{U''\left(c^0_{i,}\right)} p_i' \gamma_i \left(\frac{\partial Y_1}{\partial \sigma}\right)^2 \\
&+ \frac{U'\left(c^e_{i,1}\right) - U'\left(c^u_{i,1}\right)}{U''\left(c^0_{i,}\right)} \left[p_i''\left(Y_1\right)\left(\frac{\partial Y_1}{\partial \sigma}\right)^2 + p_i'\left(Y_1\right) \frac{\partial^2 Y_1}{\partial \sigma^2}\right] + \ldots
\end{aligned}$$

$$\frac{\left.\frac{d^2 c_{i,0}}{d\sigma^2}\right|_{\sigma=0}}{\beta \text{RMPS}_{i,0}} = \frac{U'''\left(c^e_{i,1}\right)}{U'''\left(c^0_{i,}\right)} \gamma_i^2 p_i \left(\frac{\partial Y_1}{\partial \sigma}\right)^2 + 2\frac{U''\left(c^e_{i,1}\right)}{U''\left(c^0_{i,}\right)} p_i' \gamma_i \left(\frac{\partial Y_1}{\partial \sigma}\right)^2$$

$$+ \frac{U'\left(c^e_{i,1}\right) - U'\left(c^u_{i,1}\right)}{U''\left(c^0_{i,}\right)} \left[p_i''\left(Y_1\right)\left(\frac{\partial Y_1}{\partial \sigma}\right)^2 + p_i'\left(Y_1\right)\frac{\partial^2 Y_1}{\partial \sigma^2}\right] + \ldots$$

- Precautionary savings motive a la Kimball (1990) to insure against wage risk $\gamma_i \frac{\partial Y_1}{\partial \sigma}$, given employment with p_i.

$$\frac{\left.\frac{d^2 c_{i,0}}{d\sigma^2}\right|_{\sigma=0}}{\beta \mathrm{RMPS}_{i,0}} = \frac{U'''\left(c^e_{i,1}\right)}{U'''\left(c^0_{i,}\right)}\gamma_i^2 p_i \left(\frac{\partial Y_1}{\partial \sigma}\right)^2 + 2\frac{U''\left(c^e_{i,1}\right)}{U''\left(c^0_{i,}\right)} p_i' \gamma_i \left(\frac{\partial Y_1}{\partial \sigma}\right)^2$$

$$+ \frac{U'\left(c^e_{i,1}\right) - U'\left(c^u_{i,1}\right)}{U''\left(c^0_{i,}\right)}\left[p_i''\left(Y_1\right)\left(\frac{\partial Y_1}{\partial \sigma}\right)^2 + p_i'\left(Y_1\right)\frac{\partial^2 Y_1}{\partial \sigma^2}\right] + \dots$$

- Unemployment (disaster) risk for the HH.
- *Partial* equilibrium effect: non linearity in the job search.

A simple 2 period economy: 4 channels for uncertainty

$$\frac{\left.\frac{d^2 c_{i,0}}{d\sigma^2}\right|_{\sigma=0}}{\beta \mathrm{RMPS}_{i,0}} = \frac{U'''\left(c^e_{i,1}\right)}{U'''\left(c^0_{i,}\right)} \gamma_i^2 p_i \left(\frac{\partial Y_1}{\partial \sigma}\right)^2 + 2\frac{U''\left(c^e_{i,1}\right)}{U''\left(c^0_{i,}\right)} p_i' \gamma_i \left(\frac{\partial Y_1}{\partial \sigma}\right)^2$$

$$+ \frac{U'\left(c^e_{i,1}\right) - U'\left(c^u_{i,1}\right)}{U''\left(c^0_{i,}\right)} \left[p_i''\left(Y_1\right)\left(\frac{\partial Y_1}{\partial \sigma}\right)^2 + p_i'\left(Y_1\right)\frac{\partial^2 Y_1}{\partial \sigma^2}\right] + \dots$$

- Unemployment (disaster) risk for the HH.
- *General* equilibrium effect: macro non linearity ⇒ decrease in aggregate output ⇒ increase in unemployment risk.

A simple 2 period economy: 4 channels for uncertainty

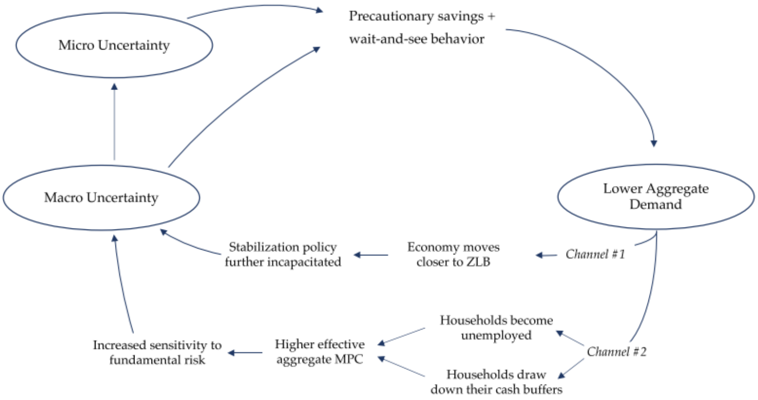
$$\frac{\left.\frac{d^2 c_{i,0}}{d\sigma^2}\right|_{\sigma=0}}{\beta \text{RMPS}_{i,0}} = \frac{U'''\left(c^e_{i,1}\right)}{U'''\left(c^0_{i,}\right)}\gamma_i^2 p_i \left(\frac{\partial Y_1}{\partial \sigma}\right)^2 + 2\frac{U''\left(c^e_{i,1}\right)}{U''\left(c^0_{i,}\right)} p_i' \gamma_i \left(\frac{\partial Y_1}{\partial \sigma}\right)^2$$

$$+ \frac{U'\left(c^e_{i,1}\right) - U'\left(c^u_{i,1}\right)}{U''\left(c^0_{i,}\right)}\left[p_i''\left(Y_1\right)\left(\frac{\partial Y_1}{\partial \sigma}\right)^2 + p_i'\left(Y_1\right)\frac{\partial^2 Y_1}{\partial \sigma^2}\right] + \dots$$

- Joint risk between *micro* uncertainty $\frac{U''\left(c^e_{i,1}\right)}{U''\left(c^0_{i,}\right)}$ and *macro* employment risk $p_i'\gamma_i \left(\frac{\partial Y_1}{\partial \sigma}\right)^2$

ECONOMIC MECHANISMS FOR THE FULL MODEL

Figure 8: General equilibrium interaction between micro, macro uncertainty and aggregate demand



HANK MODEL

- Heterogeneous agent New Keynesian model, as in Kaplan, Moll, Violante (2018)
- Main aggregate shock: discount rate AR(1) process:

$$d\rho_t = \theta_\rho(\bar{\rho} - \rho)dt + \sigma_\rho dB_t$$

HANK Model

- Heterogeneous agent New Keynesian model, as in Kaplan, Moll, Violante (2018)
- Main aggregate shock: discount rate AR(1) process:

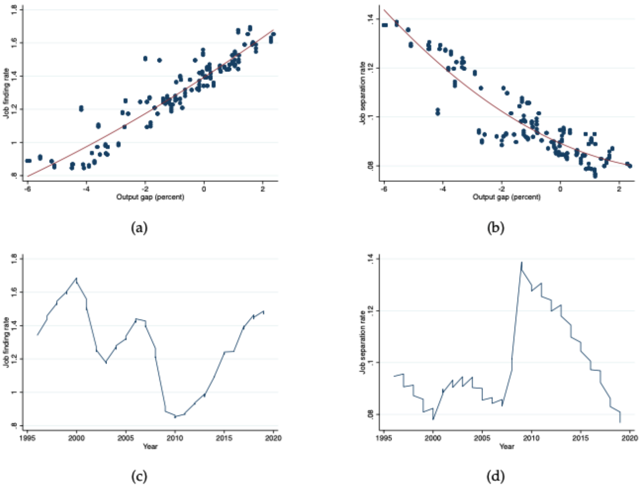
$$d\rho_t = \theta_\rho(\bar{\rho} - \rho)dt + \sigma_\rho dB_t$$

- Heterogenous Household block, with :
  - Unemployment risk (micro uncertainty) depending on the macro state
  - Two assets: Liquid asset (cash and flow of income) and Illiquid asset (capital, with adj. cost)
- Monopolistic competitive (intermediate) firms selling to CES retailler. Rotemberg price adjustment
- Representative Capital producer (q theory)
- Labor demand from Labor Unions with nominal wage stickiness
- Monetary policy: Taylor rule with ZLB
- Government with distortionary tax and U.I.

Heterogenous Household block: Micro uncertainty

- Uninsurable earning risk in z^j (or simply subscript j) a two-states Markov Process: $z_t \in \{z^E, z^U\}$ with intensity λ_t^j
- λ_t^U the job finding (intensity) rate and λ_t^E the job separation rate are "state-dependent" on changes in aggregate activity Y_t
- Transitions represented as a reduce form $\lambda_t^j = \lambda^j(y_t) = a_0 + a_1 y_t + a_2 y_t^2 + \dots$
- Estimated with the Current Population Survey (CPS)

Figure 2: Employment Transition Rates over the Business Cycle



Job finding rate :

$$\lambda_t^U = \underset{(114.96)}{1.39} + \underset{(13.66)}{0.115}\, y_t + \underset{(1.08)}{0.0026} y_t^2 + \dots$$

Job separation rate :

$$\lambda_t^E = \underset{(88.67)}{0.89} - \underset{(-6.85)}{0.0053} y_t + \underset{(3.62)}{0.0006} y_t^2 + \dots$$

Heterogenous Household block: two assets models

- Households discount future at rate ρ_t, die at rate ζ, consumes c_t and supply labor h_t, and owns two assets:
- A liquid asset position evolves according to

$$\dot{a}_t = s_t^j(a,k) = (r_t + \zeta)\, a_t + k_t \frac{dR_t}{dt} + e_t - q_t \iota_t - \psi\left(\iota_t, k_t\right) - c_t$$

- Earning e_t collecting wages and rebates:

$$e_t = \left(1 - \tau^{\text{lab}}\right) z_t w_t h_t + \tau_t^{\text{lump}} + \tau^{\text{UI}}\left(z_t\right)$$

Heterogenous Household block: two assets models

- Households discount future at rate ρ_t, die at rate ζ, consumes c_t and supply labor h_t, and owns two assets:
- A liquid asset position evolves according to

$$\dot{a}_t = s_t^j(a,k) = (r_t + \zeta)\, a_t + k_t \frac{dR_t}{dt} + e_t - q_t \iota_t - \psi\left(\iota_t, k_t\right) - c_t$$

- Earning e_t collecting wages and rebates:

$$e_t = \left(1 - \tau^{\text{lab}}\right) z_t w_t h_t + \tau_t^{\text{lump}} + \tau^{\text{UI}}\left(z_t\right)$$

- Illiquid asset position as usual LoM for capital $\dot{k}_t = (\zeta - \delta)k_t + \iota_t = m_t^j(a,k)$
- Borrowing constraint on liquid asset $a_t > 0$ and short selling constraint on capital $k_t > 0$
- Investment adjustment cost $\psi(\iota_t, k_t) = \psi_0|\iota_t| + \psi_1\left(\frac{\iota_t}{k_t}\right)^2 k_t$

- ***Retailer*** aggregating varieties $Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon^f - 1}{\epsilon^f}} dj \right)^{\frac{\epsilon^f}{\epsilon^f - 1}}$ yielding intermediate inputs demand: $Y_t(j) = (P_t(j)/P_t)^{-\epsilon^f} Y_t$, for $P_t(j)$ price of firm j

FIRM BLOCK(S)

- ***Retailer*** aggregating varieties $Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon^f - 1}{\epsilon^f}} dj\right)^{\frac{\epsilon^f}{\epsilon^f - 1}}$ yielding intermediate inputs demand: $Y_t(j) = (P_t(j)/P_t)^{-\epsilon^f} Y_t$, for $P_t(j)$ price of firm j
- ***Firm*** j's production function is given by $Y_t(j) = K_t(j)^{1-\beta} L_t(j)^{\beta}$ with β labor share, $L_t(j)$ labor demand and W_t nominal wage rate
- Rental market for capital from households with nominal rental rate i_t^k and $r_t^k = i_t^k / P_t$, where $r_t^k = MPK$

- ***Retailer*** aggregating varieties $Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon^f - 1}{f}} dj \right)^{\frac{\epsilon f}{\epsilon^f - 1}}$ yielding intermediate inputs demand: $Y_t(j) = (P_t(j)/P_t)^{-\epsilon^f} Y_t$, for $P_t(j)$ price of firm j
- ***Firm*** j's production function is given by $Y_t(j) = K_t(j)^{1-\beta} L_t(j)^{\beta}$ with β labor share, $L_t(j)$ labor demand and W_t nominal wage rate
- Rental market for capital from households with nominal rental rate i_t^k and $r_t^k = i_t^k / P_t$, where $r_t^k = MPK$
- Dynamic price setting as in Rotemberg:

$$\max_{\{\pi_t(j)\}} \mathbb{E}_0 \int_0^{\infty} e^{-\int_0^t i_s^k ds} \left[(1 - mc_t) P_t(j) Y_t(j) - \Lambda(\pi_t(j)) \right] dt$$

with mc_t real marginal cost and Λ quadratic cost in price adjustment. This gives rise to a HJB equation

- ***Capital producer***: create capital with adjustment cost $\Phi(I_t/K_t)$
- Profit and capital price:

$$\Pi_t^Q = q_t I_t - I_t - \Phi\left(\frac{I_t}{K_t}\right) K_t \qquad\qquad q_t = 1 + \Phi'\left(\frac{I_t}{K_t}\right)$$

- No arbitrage yields the return on capital in HH wealth.

$$dR_t = (r^k + \frac{\Pi_t^Q}{K_t})dt$$

- ***Capital producer***: create capital with adjustment cost $\Phi(I_t/K_t)$
- Profit and capital price:

$$\Pi_t^Q = q_t I_t - I_t - \Phi\left(\frac{I_t}{K_t}\right) K_t \qquad\qquad q_t = 1 + \Phi'\left(\frac{I_t}{K_t}\right)$$

- No arbitrage yields the return on capital in HH wealth.

$$dR_t = (r^k + \frac{\Pi_t^Q}{K_t})dt$$

- ***Labor unions*** aggregate labor varieties $L_t = \left(\int L_{k,t}^{\frac{\epsilon^w - 1}{\epsilon^w}} dk\right)^{\frac{\epsilon^w}{\epsilon^w - 1}}$ and the dynamic wage setting given by

$$\max_{\pi_{k,t}^w} \mathbb{E}_0 \int_0^\infty e^{-\int_0^t (\rho_s + \zeta) ds} \left[\int u\left(c_t, h_t\right) g_t d(a,k,z) - \frac{\chi^w}{2} \left(\pi_{k,t}^w\right)^2 L_t\right] dt$$

This gives rise to another HJB equation

- Monetary policy: simple Taylor rule and subject to zero lower bound (ZLB)

$$i_t = \max\{r^* + \bar{\pi} + \lambda_\pi \pi_t + \lambda_Y y_t, 0\}$$

- Government collects taxes, pays interests on debt B^G and gives the rest lump-sum or as Unemployment insurance
- Market clearing: 3 markets: liquid assets, goods, investment

$$A_t = \int a g_t(a,k,z) d(a,k,z) = B^G$$
$$Y_t = C_t + I_t + \Phi_t + \Psi_t + G_t$$
$$I_t \int \iota_t(a,k,z) g_t(a,k,z) d(a,k,z)$$

with $C_t = \int c_t(a,k,z) g_t(a,k,z) d(a,k,z)$ and analogously for I_t and Ψ_t

Solution method : Overview

- State variable of the model: idiosyncratic HH states: $x = (a, k, z^j)$ and aggregate states $\Gamma_t = (\rho_t, g_t)$
- The usual method dealing with HA models is to solve a system of (coupled) PDEs: the Hamilton Jacobi Bellman for $V(x)$ and the Kolmogorov Forward for $g(x)$.

- State variable of the model: idiosyncratic HH states: $x = (a, k, z^j)$ and aggregate states $\Gamma_t = (\rho_t, g_t)$
- The usual method dealing with HA models is to solve a system of (coupled) PDEs: the Hamilton Jacobi Bellman for $V(x)$ and the Kolmogorov Forward for $g(x)$.
- However, with aggregate shocks, the PDE system becomes stochastic (i.e. varies w/ dB_t)
- The approach of Schaab, following Cardaliaguet, Lions, Lasry, Delarue (2018) is to focus on the master equation: an infinite dimensional equation for $\mathcal{V}(x, \Gamma)$
- This includes the effect of the distribution on the value: $\frac{\delta \mathcal{V}}{\delta g}$

Solution method : Overview

- State variable of the model: idiosyncratic HH states: $x = (a, k, z^j)$ and aggregate states $\Gamma_t = (\rho_t, g_t)$
- The usual method dealing with HA models is to solve a system of (coupled) PDEs: the Hamilton Jacobi Bellman for $V(x)$ and the Kolmogorov Forward for $g(x)$.
- However, with aggregate shocks, the PDE system becomes stochastic (i.e. varies w/ dB_t)
- The approach of Schaab, following Cardaliaguet, Lions, Lasry, Delarue (2018) is to focus on the master equation: an infinite dimensional equation for $\mathcal{V}(x, \Gamma)$
- This includes the effect of the distribution on the value: $\frac{\delta \mathcal{V}}{\delta g}$
- The idea is to use this master equation with a finite dimensional representation for $g_t(x)$

$$\hat{g}_t(x) = F(\alpha_t)(x) \approx g_t(x)$$

- State variable of the model: idiosyncratic HH states: (a, k, z^j) and aggregate states $\Gamma_t = g_t$
- The dynamic of the distribution of agents : $g(a, k, z_j) = g^j(a, k)$ without aggregate shocks

$$
\begin{aligned}
\frac{dg^j(a,k)}{dt} &= \mathcal{A}^* g^j(a,k) \\
&= -\partial_a \left[s^j(a,k,\Gamma) g^j(a,k) \right] - \partial_k \left[m^j(a,k,\Gamma) g^j(a,k) \right] - \lambda^j(\Gamma) g^j(a,k) + \lambda^{-j}(\Gamma) g^{-j}(a,k)
\end{aligned}
$$

- The Master equation without aggregate shocks with states (a, k, z^j) and aggregate states $\Gamma_t = g_t$

$$
\begin{aligned}
(\rho + \zeta) V^j(a,k,\Gamma) = & \max_{c,h,j} \left\{ u\left(c^j, h^j\right) + s^j \partial_a V^j(a,k,\Gamma) + m^j \partial_k V^j(a,k,\Gamma) \right\} \\
& + \lambda^j(\Gamma) \left[V^{-j}(a,k,\Gamma) - V^j(a,k,\Gamma) \right] \\
& + \sum_l \int \frac{\delta V^l}{\delta g} \left(\mathcal{A}^* g^l \right)(a,k) d(a,k)
\end{aligned}
$$

- The Master equation without aggregate shocks with states (a, k, z^j) and aggregate states $\Gamma_t = g_t$

$$(\rho+\zeta)V^j(a,k,\Gamma) = \max_{c,h,j}\left\{u\left(c^j,h^j\right) + s^j\partial_a V^j(a,k,\Gamma) + m^j\partial_k V^j(a,k,\Gamma)\right\}$$
$$+ \lambda^j(\Gamma)\left[V^{-j}(a,k,\Gamma) - V^j(a,k,\Gamma)\right]$$
$$+ \underbrace{\sum_l \int \frac{\delta V^l}{\delta g}\left(\mathcal{A}^* g^l\right)(a,k)\, d(a,k)}_{\text{Effect of changes of the distribution on HH value}}$$

- The dynamic of the distribution of agents : $g(a,k,z_j,\rho) = g^j(a,k,\rho)$ with aggregate shocks

$$
\begin{aligned}
\frac{dg^j(a,k,\rho)}{dt} &= \mathcal{A}^* g^j(a,k,\rho) dt + \mathcal{B}^* g^j(a,k,\rho) dB_t \\
&= -\partial_a \left[s^j(a,k,\Gamma) g^j(a,k,\rho) \right] - \partial_k \left[m^j(a,k,\Gamma) g^j(a,k,\rho)(a,k) \right] \\
&\quad - \lambda^j(\Gamma) g^j(a,k,\rho) + \lambda^{-j}(\Gamma) g^{-j}(a,k,\rho) - \partial_\rho \left[\theta_\rho (\bar{\rho} - \rho) \partial_\rho g^j(a,k,\rho) \right] \\
&\quad + \frac{\sigma_\rho^2}{2} \partial_{\rho\rho} g^j(a,k,\rho) - \partial_\rho \left[\sigma_\rho g^j(a,k,\rho) \right] dB_t
\end{aligned}
$$

- The dynamic of the distribution of agents : $g(a,k,z_j,\rho) = g^j(a,k,\rho)$ with aggregate shocks

$$\begin{aligned}\frac{dg^j(a,k,\rho)}{dt} &= \mathcal{A}^* g^j(a,k,\rho)dt + \mathcal{B}^* g^j(a,k,\rho)dB_t \\ &= -\partial_a\left[s^j(a,k,\Gamma)g^j(a,k,\rho)\right] - \partial_k\left[m^j(a,k,\Gamma)g^j(a,k,\rho)(a,k)\right] \\ &\quad - \lambda^j(\Gamma)g^j(a,k,\rho) + \lambda^{-j}(\Gamma)g^{-j}(a,k,\rho) \underbrace{-\partial_\rho\left[\theta_\rho(\bar{\rho}-\rho)\partial_\rho g^j(a,k,\rho)\right] + \frac{\sigma_\rho^2}{2}\partial_{\rho\rho}g^j(a,k,\rho) - \partial_\rho\left[\sigma_\rho g^j(a,k,\rho)\right]dB_t}_{\text{Effect of aggregate shocks}}\end{aligned}$$

- The Master equation with aggregate shocks with states $x = (a, k, z^j)$ and aggregate states $\Gamma_t = (\rho_t, g_t)$

$$
\begin{aligned}
(\rho+\zeta)V^j(a,k,\Gamma) = \max_{c,h,j} & \left\{u\left(c^j, h^j\right) + s^j \partial_a V^j(a,k,\Gamma) + m^j \partial_k V^j(a,k,\Gamma)\right\} \\
& + \lambda^j(\Gamma)\left[V^{-j}(a,k,\Gamma) - V^j(a,k,\Gamma)\right] + \theta_\rho(\bar{\rho}-\rho)\,\partial_\rho V^j(a,k,\Gamma) \\
& + \frac{\sigma_\rho^2}{2}\partial_{\rho\rho} V^j(a,k,\Gamma) + \sum_l \int \frac{\delta V^l}{\delta g}\left(\mathcal{A}^* g^l\right)(a,k,\rho) d(a,k,\rho) \\
& + \sum_l \int \partial_\rho \left[\frac{\delta V^l(a,k,\Gamma)}{\delta g}\right] d(a,k,\rho) + \frac{\sigma_\rho^2}{2} \iint \partial_{\rho\rho} \frac{\delta^2 V^l(a,k,\Gamma)}{\delta g^2} d(a,k,\rho)^2
\end{aligned}
$$

- The Master equation with aggregate shocks with states $x = (a, k, z^j)$ and aggregate states $\Gamma_t = (\rho_t, g_t)$

$$
\begin{aligned}
(\rho+\zeta)V^j(a,k,\Gamma) = \max_{c,h,j} & \left\{ u\left(c^j, h^j\right) + s^j \partial_a V^j(a,k,\Gamma) + m^j \partial_k V^j(a,k,\Gamma) \right\} \\
& + \lambda^j(\Gamma)\left[V^{-j}(a,k,\Gamma) - V^j(a,k,\Gamma)\right] + \theta_\rho(\bar{\rho}-\rho)\,\partial_\rho V^j(a,k,\Gamma) \\
& + \frac{\sigma_\rho^2}{2}\partial_{\rho\rho}V^j(a,k,\Gamma) + \sum_l \int \frac{\delta V^l}{\delta g}\left(\mathcal{A}^* g^l\right)(a,k,\rho)\, d(a,k,\rho) \\
& + \sum_l \int \partial_\rho \left[\frac{\delta V^l(a,k,\Gamma)}{\delta g}\right] d(a,k,\rho) + \frac{\sigma_\rho^2}{2} \iint \partial_{\rho\rho} \frac{\delta^2 V^l(a,k,\Gamma)}{\delta g^2} d(a,k,\rho)^2
\end{aligned}
$$

MASTER EQUATION WITH AGG. SHOCKS

- The Master equation with aggregate shocks with states $x = (a, k, z^j)$ and aggregate states $\Gamma_t = (\rho_t, g_t)$

$$
\begin{aligned}
(\rho+\zeta)V^j(a,k,\Gamma) = \max_{c,h,j} &\left\{u\left(c^j, h^j\right) + s^j\partial_a V^j(a,k,\Gamma) + m^j\partial_k V^j(a,k,\Gamma)\right\} \\
&+ \lambda^j(\Gamma)\left[V^{-j}(a,k,\Gamma) - V^j(a,k,\Gamma)\right] + \theta_\rho(\bar{\rho}-\rho)\,\partial_\rho V^j(a,k,\Gamma) \\
&+ \frac{\sigma_\rho^2}{2}\partial_{\rho\rho}V^j(a,k,\Gamma) + \sum_l \int \frac{\delta V^l}{\delta g}\left(\mathcal{A}^* g^l\right)(a,k,\rho)\,d(a,k,\rho) \\
&+ \underbrace{\sum_l \int \partial_\rho\left[\frac{\delta V^l(a,k,\Gamma)}{\delta g}\right]d(a,k,\rho) + \frac{\sigma_\rho^2}{2}\iint \partial_{\rho\rho}\frac{\delta^2 V^l(a,k,\Gamma)}{\delta g^2}d(a,k,\rho)^2}_{\text{Second order terms with agg. shocks}}
\end{aligned}
$$

- The Master equation with aggregate shocks with states $x = (a, k, z^j)$ and aggregate states $\Gamma_t = (\rho_t, g_t)$

$$
\begin{aligned}
(\rho + \zeta) V^j(a,k,\Gamma) = \max_{c,h,j} & \left\{ u\left(c^j, h^j\right) + s^j \partial_a V^j(a,k,\Gamma) + m^j \partial_k V^j(a,k,\Gamma) \right\} \\
& + \lambda^j(\Gamma) \left[V^{-j}(a,k,\Gamma) - V^j(a,k,\Gamma) \right] + \theta_\rho(\bar{\rho} - \rho)\, \partial_\rho V^j(a,k,\Gamma) \\
& + \frac{\sigma_\rho^2}{2} \partial_{\rho\rho} V^j(a,k,\Gamma) + \sum_l \int \frac{\delta V^l}{\delta g} \left(\mathcal{A}^* g^l\right)(a,k,\rho) d(a,k,\rho) \\
& + \sum_l \int \partial_\rho \left[\frac{\delta V^l(a,k,\Gamma)}{\delta g} \right] d(a,k,\rho) + \frac{\sigma_\rho^2}{2} \iint \partial_{\rho\rho} \frac{\delta^2 V^l(a,k,\Gamma)}{\delta g^2} d(a,k,\rho)^2
\end{aligned}
$$

Master equation: dimensionality reduction

- Approximate cross sectional distribution: $g\hat{(x)} = F(\alpha)(x)$, and in practice in the baseline model:

$$F(\alpha)(x) = g_0(x) + \sum_n \alpha_t^n T^n(x)$$

- This approximation makes the model/master equation tractable, with approximate agg. state $\hat{\Gamma} = (x, \hat{g})$ and value :

$$V(x, \rho, g) \approx V(x, \rho, \hat{g}) = \hat{V}(x, \rho, \alpha)$$

- The household only have to track the law of motion of α:

$$d\alpha_t = \mu_\alpha(\hat{\Gamma})dt + \sigma_\alpha(\hat{\Gamma})dB_t$$

Approximate Master equation

- The HJB/Master equation with aggregate shocks with states $x = (a, k, z^j)$ and aggregate states $\hat{\Gamma}_t = (\rho_t, \alpha_t)$

$$
\begin{aligned}
(\rho+\zeta)V^j(a,k,\hat{\Gamma}) = \max_{c,h,j} \Big\{ & u\left(c^j, h^j\right) + s^j \partial_a V^j(a,k,\hat{\Gamma}) + m^j \partial_k V^j(a,k,\hat{\Gamma}) \Big\} \\
& + \lambda^j(\hat{\Gamma}) \left[V^{-j}(a,k,\hat{\Gamma}) - V^j(a,k,\hat{\Gamma}) \right] + \theta_\rho(\bar{\rho}-\rho)\, \partial_\rho V^j(a,k,\hat{\Gamma}) \\
& + \frac{\sigma_\rho^2}{2} \partial_{\rho\rho} V^j(a,k,\hat{\Gamma}) + \mu_\alpha(\hat{\Gamma}) \partial_\alpha V^j(a,k,\hat{\Gamma}) + \sigma_\alpha(\hat{\Gamma})^T \partial_{\alpha\alpha} V^j(a,k,\hat{\Gamma}) \sigma_\alpha(\hat{\Gamma}) \\
& + \sigma_\alpha(\hat{\Gamma})^T \partial_{\alpha\rho} V^j(a,k,\hat{\Gamma}) \sigma_\rho + \sigma_\rho \partial_{\rho\alpha} V^j(a,k,\hat{\Gamma}) \sigma_\alpha(\hat{\Gamma})^T + \text{second order terms}
\end{aligned}
$$

- Some notations:
  - x: the vector for (a,k,z); N_x: the number of grid points on x; N_α: the number of grid points for distribution.
- Key tricks:
  - For KFE: Approximate the distribution using basis functions to reduce the dimensionality
    - (from $> N_x$ dimensions to 8-10 dimensions in the benchmark economy)
  - For HJB: Use adaptive sparse grid to further reduces grid points
    - 340 points over the (a, k) dimensions vs. 4,200
    - Mostly covered in another paper Schaab and Zhang (2020)

- Concept of an equilibrium solution:
  - "Micro" functions $\{V, g, c, \iota\}(x, \Gamma)$
  - Macro functions $\{r, r^k, q, H, ...\}(\Gamma)$

Algorithm Structure:

- Level 3: Update the distribution to minimize forecast errors (Similar to Krusell and Smith, 1998)
  - Level 2: Solve for GE prices to clear markets
    - Level 1: Given prices, solve the household's master equation which give policy functions and stationary distribution (based on Achdou et al., 2017)

Consider the distribution in the affine form: $F(\alpha_t)(x) = g^0(x) + \sum_i \alpha_t^i T^i(x)$, where

- $g^0(x)$ is the distribution at deterministic steady state, i.e., $\lim_{\sigma_\rho \to 0} g(x; \sigma_\rho)$
- $T^i(x) : \mathbb{R}^{N_x} \to \mathbb{R}$ basis functions:
  - Parametrically: Fixed as some parametric functions
    - Linear basis
    - Chebyshev polynomials
  - Non-parametrically: Update $T(x)$ based on errors (more on this later)
- $\alpha_t^i \in \mathbb{R}^{N_\alpha}$ the time-varying weights on $T^i(x)$

Dynamics of α_t

Consider the case where the distribution is locally deterministic (i.e., no dB term in $d\alpha_t$)

- From KFE, we have:

$$d\hat{g}_t(x) = \left(\mathcal{A}^*\hat{g}_t\right)(x)dt$$

  - This equation can be evaluated for every point on the grid of x.
- Using our representation of $g(x)$, we have $d\hat{g}_t(x) = dF(\alpha_t)(x) = F_\alpha(\alpha_t)(x)\mu_\alpha(\Gamma)dt$
- By matching the coefficient, we have:

$$\underbrace{F_\alpha(\alpha_t)(x)}_{N_x \times N_\alpha}\underbrace{\mu_\alpha(\Gamma)}_{N_\alpha \times 1} = \underbrace{\left(\mathcal{A}^*F(\alpha_t)\right)(x)}_{N_x \times 1}$$

- To solve μ_α, we take the pseudo-inverse (OLS):

$$\mu_\alpha = \left(F_\alpha^T F_\alpha\right)^{-1} F_\alpha^T \left(\mathcal{A}^*F\right)$$

A trivial example

- Consider the case where the distribution is approximated on the same grid for x using linear basis, i.e.,

$$F(x;\alpha_t) = \sum_i \alpha_{i,t} \mathbf{1}_{\{x_{i-1} < x \leq x_i\}}$$

Then $F_\alpha(x)$ evaluated on the x-grid is exactly the identity matrix. Then the drift of μ_α according to the formula above is exactly

$$\mu_a = \mathcal{A}^* F$$

- When $N_\alpha < N_x$, the pseudo-inverse provides an efficient way to approximate the larger grid with a small number of basis.

Choice of parametric or non-parametric representation

- Parametric:
  - With $T(x)$ chosen ex-ante and $\mu(\alpha)$ computed above, we can solve the equilibrium without the level-3 loop
  - Efficient for simple models e.g. one-asset HANK, Krusell-Smith
- Non-parametric:
  - Need additional steps with simulation to pin down the optimal $T(x)$
  - Achieve the same accuracy with far fewer dimensions N_a
  - Used for two-asset HANK

Non-parametric steps (Level 3)

- Key idea: Update $T(x)$ using simulations from a finer grid.
- Steps:
  1. Given a $T(x)$ either from previous iteration or initialized from parametric steps.
  2. Solve the equilibrium following level 1 and level2 as before;
  3. Simulate the household sector from $\hat{g}^0(x)$, hit the economy with shocks and update $\hat{g}_t^{lom}(x)$ using law of motion (μ_α)
  4. Simulate the household on a finer grid (e.g. the x-grid), fixing $V(x,\Gamma)$, solving the static equilibrium period by period (costly!) . Update the distribution using the real policy (KFE)
  5. Compute the error $\left\| g_t^{n,\text{lom}}(x) - g_t^{n,\text{sim}}(x) \right\|_{(t,x)\in\mathbb{R}\times\mathbb{R}^d}$, if not converged, update $T(x)$ (next slide).

UPDATING $T(x)$

Within the affine distribution family, we are looking for α_t and $T(x)$ to minimize the error:

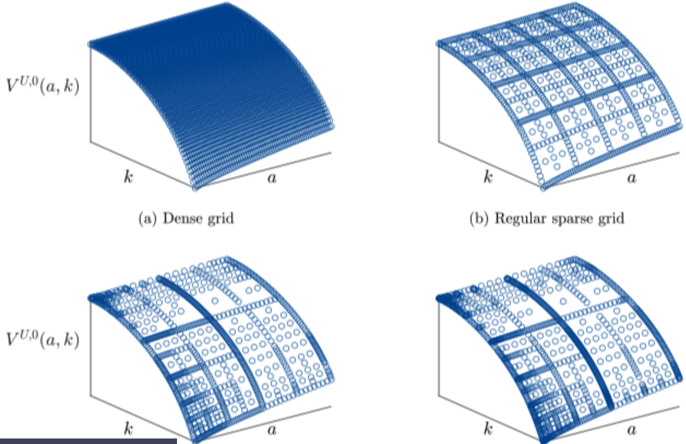
$$\min_{T(x),\alpha_t} \left\| g_t^{sim}(x) - g^0(x) - \alpha_t T(x) \right\|_{\mathbb{L}^2(t \times x)}$$

It can be shown that the optimal α_t and $T(x)$ is given as:

$$\alpha_t' = \left(T(x)T(x)'\right)^{-1} F(x)(g_t^{sim}(x) - g^0(x))'$$

$$T(x) = \left(\sum_t \alpha_t' \alpha_t\right)^{-1} \sum_t \alpha_t' \left[g_t^{sim}(x) - g^0(x)\right]$$

Reference is in Schaab and Zhang (2020), no where to be found. It's said to be based on the discrete version by Brumm and Scheidegger (2017).



(a) Dense grid

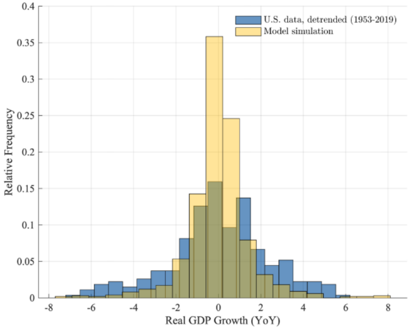
(b) Regular sparse grid

General comments on the algorithm

- Adaptive sparse grid and distribution representation seem promising for higher dimension problems;
- Seems still challenging to deal with stochastic KFE (the cross-derivative term?).
- More theoretical refinement required - The current stage looks like a kludge that learned from trial and error.

SIMULATION: A QUALITATIVE FIT OF THE MAIN MOMENTS

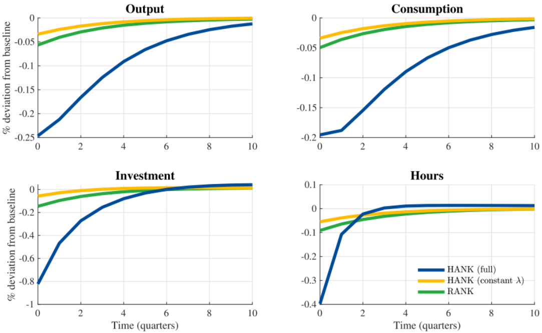
Figure 3: Distribution of GDP growth in U.S. postwar data and model simulations



Notes. Relative frequency histogram of distribution of year-over-year GDP growth in U.S. postwar data since 1953 (blue bars) and model simulations (yellow bars). Data is detrended using a Kalman filter.

Comparison with HANK-RANK

Figure 5: Impulse responses of business cycle aggregates to fundamental risk shock, $\Delta\sigma_\rho$



Notes. Comparison of impulse responses of business cycle aggregates to fundamental risk shock across three models. The shock has a half-life of 1 quarter and is initialized at half the size of the Basu and Bundick (2017) shock to facilitate comparison. The blue, yellow and green lines correspond, respectively, to the baseline model, the model that shuts off the interaction between micro and macro uncertainty, and the associated RANK model.

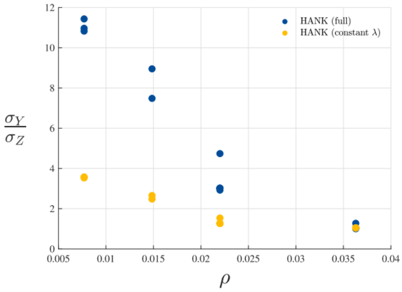
Comparison with HANK-RANK: effects decomposition

Table 2: Decomposition of the effect of fundamental risk shock $\Delta\sigma_\rho$ on aggregate consumption

	HANK		RANK
Contribution to % change in C_0	*Normal times*	*Crisis region*	*Normal times*
Direct effect: uncertainty (micro and macro)	−0.19	−0.22	−0.05
Indirect effect: micro uncertainty	−0.47	−0.51	0.00
Indirect effect: disposable income	0.58	0.66	0.02
Indirect effect: portfolio returns	0.04	−0.12	−0.03
Other effects	−0.16	−0.12	0.01
Total effect (% change in C_0)	−0.2	−0.31	−0.05

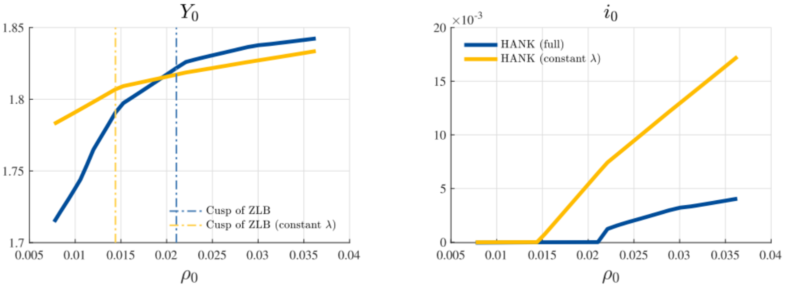
Uncertainty multiplier, micro uncertainty, and aggregate state

Figure 10: State space representation of Uncertainty Multiplier for different values of discount rate ρ



Notes. Comparison of Uncertainty Multiplier $G_Y(\rho,\alpha) = \frac{\sigma_Y(\rho,\alpha)}{\sigma_\rho}$ across different models via state space representation. The x-axis traces out different values in ρ, and the y-axis maps these into $G_Y(\rho,\alpha)$. For a given ρ, vertically aligned dots correspond to different values of α which are added for illustration. The blue and yellow dots correspond, respectively, to the baseline model and the model that shuts off the interaction between micro and macro uncertainty.

Figure 11: Anticipation effects give rise to "paradox of prudence" near cusp of ZLB



Notes. Panels trace out on-impact response in output and the nominal interest rate (y-axis) for simulations that initialize the economy at different shock realizations ρ_0 (x-axis). Blue and yellow lines correspond, respectively, to the baseline model and the model that shuts off the interaction between micro and macro uncertainty.

References

Kaplan, Greg, Benjamin Moll, and Giovanni L Violante (2018). "Monetary policy according to HANK". In: *American Economic Review* 108.3, pp. 697–743.

Kimball, Miles S (1990). *Precautionary saving and the marginal propensity to consume*. Tech. rep. National Bureau of Economic Research.