Macroeconomics with Learning and Misspecifications
A General Theory and Applications

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Macro Reading Group

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  - However, abandoning rational expectations leads a modeler into the ”wilderness of bounded rationality” (Sargent)
  - Recent profusion of different behavioral models
- This paper proposes a unified and flexible framework of behavioral macroeconomics
  - Build a theory where agents face constraints on beliefs & behaviors
  - Agents’ model of the economy is different from the one they live in :
    - Their model is ”misspecified”
  - They can learn about the economy and take optimal decisions

Introduction – Contribution

⇒ Three contributions of the paper :

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   - Agents select their belief from a constrained set
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   - Concept of *"Constrained-rational expectation equilibrium"* (CREE), where rational expectation (REE) is a special case

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   - Foundation as the limit of Bayesian/Adaptative learning
   - Concept of *"Constrained-rational expectation equilibrium"* (CREE), where rational expectation (REE) is a special case

2. New deviation from REE with low-dimensional (linear) hidden factor models
   - Conditions for amplification and persistence of shocks
3. Applications to medium scale DSGE, for quantitative results.

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     - Measure of *distance* between the beliefs $Q_\theta(dy)$ and reality of observables $T(dy)$
  3. Constrained rational-expectations equilibrium (CRRE) as fixed-point (invariant distribution) of steps 1. and 2.

Model – Step 1 – Recursive Temporary equilibrium T

- Agents make choices x_t, see observables y_t
- *States* z_t are unobserved and follow $z_t \sim \Pi(\cdot|y_{t-1}, z_{t-1})$
- Agent believe $y_t \sim Q_\theta(\cdot|y_{t-1})$, i.e. subjective model of transitions

$$V(x_-, y, \theta) = \max_{x \in \Gamma(x_-, y)} J(x_-, x, y') = \max_{x \in \Gamma(x_-, y)} \left[u(x_-, x, y) + \beta \int_Y V(x, y', \theta) Q_\theta(dy'|y) \right]$$

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- A recursive temporary equilibrium is a transition probability $T : (x_-, y_-, z_-, \theta) \mapsto T(x, y, z|x_-, y_-, z_-, \theta) \in \mathcal{P}(X \times Y \times Z)$, s.t. :
  (i) Agents decisions are optimal :
  $$x \in \boldsymbol{x}(x_-, x, y') = \underset{x \in \Gamma(x_-,y)}{\operatorname{argmax}} J(x_-, x, y') \qquad \text{for } T\text{-a.e.}(x, y)$$
  (ii) General-equilibrium requirement are satisfied :
  $$G(x, y, z) = 0,$$
  (iii) Consistency between transition of state and equilibrium outcomes :
  $$T(B|x_-, y_-, z_-, \theta) = \Pi(B|y_-, z_-) \qquad \forall B \subset Z$$

Model – Step 2 – Set of beliefs / Kullback-Leibner div.

- Parameters of agents' belief are included in a (potentially constrained) *set of models* $\theta \in \Theta$
  - Choose the model – used for Q_θ (density q_θ) – to be the closest to the distribution of observables, i.e. $T(x, y, z | x_-, y_-, z_-, \theta)$
- Imply to minimize the Kullback-Leibner divergence, i.e. for any μ

$$H(Q_\theta, T, \mu) = -\int_{X \times Y \times Z} \int_Y \log\left(q_\theta(y|y_-)\right) T(dy|x_-, y_-, z_-, \theta)\, \mu(dx_- \times dy_- \times dz_-)$$

  - If $Q_\theta(dy) \equiv T(dy|\cdot)$ μ-a.e. then $H(Q_\theta, T, \mu) = 0$
  - Θ may not contains the model generating T, in such case, the agents' model is *misspecified*
  - Bayesian/Adaptative learning foundation for this distance

Model – Step 3 – CRRE

- For a given temporary equilibrium T, a constrained rational-expectations equilibrium (CRRE) is a triple (T, μ^*, Θ^*)s.t. :
  (i) There is a fixed point of the temporary equilibrium, i.e. μ^* is the stationary distribution of T, (i.e. $\mu^* = \mu^* T$)
  (ii) The agent optimally choose its best model : $\Theta^* = \text{argmax}_{\theta \in \Theta} H(Q_\theta, T, \mu^*)$
  (iii) μ^* is supported on $X \times Y \times Z \times \Theta^*$
- **Thm 1 :** If the economy has a continuous temporary eq. T, then a CRRE exists
- *Proof :* Kakutani-Glicksberg-Fan fixed point theorem.

Learning foundations for CREE

- Bayesian and Adaptative learning equilibria as foundation for the minimization of K-L-div.
- Bayesian equilibrium :

$$\mathbb{P}\left[(x_t, y_t, z_t, \lambda_t) \in B \,|\, \{x_s, y_s, z_s, \lambda_s\}_{s=0}^{t-1}\right] = T(B|x_{t-1}, y_{t-1}, z_{t-1}, \theta), \qquad \forall B$$

- Asymptotic Mean stationarity : with $\mu_t(B) = \mathbb{P}\left((x_t, y_t, z_t, \theta) \in B\right)$,

$$\overline{\mu_t}(B) = \frac{1}{t}\sum_{s=0}^{t-1} \mu_s(B) \xrightarrow[t\to\infty]{weakly} \overline{\mu}(B)$$

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- **Thm 2 :** For T a continuous temporary eq., if it is asymptotic mean stationary then there is a proba distribution μ^* and Θ^* closed set s.t. :
  (i) For $U_1 \subset K \subset U_2$, we have $\mu^*(U_1) \leq \overline{\mu_t}(B) \leq \mu^*(U_2)$
  (ii) $U \supset \Theta^*$ then $\lim_{t\to\infty} \overline{\lambda_t}(U) = 1$
  (iii) The triple (T, μ^*, Θ^*) is a CREE
  - *Proof :* LLN for Markov chains and extension of stat-result on concentration of Bayes estimates on minimizers of K-L-div.

Hidden factor model

- Agent subjective model of the economy (like a filtering problem) :
  - Underlying state : $\omega_t = A\omega_{t-1} + \epsilon_{\omega t}$
  - Observables : $o_t = B'\omega_t + \epsilon_{ot}$
  - Estimates (Kalman updating) : $\hat{\omega}_t = (A - KB')\hat{\omega}_{t-1} + Ko_t$
  - Parameters : $\theta = (A, B, \sigma_\omega, \sigma_o)$

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- Real model of the economy, i.e T temporary eq. :
  - Optimal choice : $x_t = (c, 1)o_t + \mathbb{E}_t^\theta \left[\sum_{s=1}^{\infty} \beta^s (c, 1)\, o_{t+s}\right]$
  - Equilibrium condition : $o_t = (x_t, z_t)'$
  - State evolution $z_t = \rho z_{t-1} + \varepsilon$

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- If rational expectations :
  - Rational estimates : $\omega_t = z_t, A = \rho, B = (1/(1 - c - \beta\rho), 1)$ etc.
  - Agents reacts : $x_t^{REE} = \frac{1}{1-c-\beta\rho} z_t$

Hidden factor model - Amplification and persistence

- Suppose the set constraint is s.t. $\Theta = \{\theta\} = \{A, B, \sigma_\omega, \sigma_o\}$ (only one model at their disposal

$$x_t^{CRRE} = \frac{1}{1 - c - \beta\varphi_{(A,B)} k_x} \left[\beta\varphi_{(A,B)} (A - KB') \hat{\omega}_{t-1} + (1 + \beta\varphi_{(A,B)} k_z)\, z_t \right]$$

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- Amplification of shocks :
  - **Prop 4 :** Depending on parameters, shocks to states (ε_t) amplify agents choice x_t under CREE.
- Persistence of shocks :
  - **Prop 5 :** Depending on parameters, shocks to states (ε_t) are history-dependent under CREE.
- Extension : with larger Θ could compute and minimize the K-L-div. and choose the

Quantitative exploration

- Medium scale DSGE model a la Christiano, Eichenbaum, Evans (2005) / Smets-Wouters (2007)
  - Agents observation and could form expectation on 14 variables in the economy
  - Only dispose of a one-factor linear model (or d for CREE-d model)
- Results :
  - Hump-shape response to TFP/Monetary shocks
  - Amplification of demand shocks (discount/gov spending) and ability to generate business cycle

Quantitative exploration

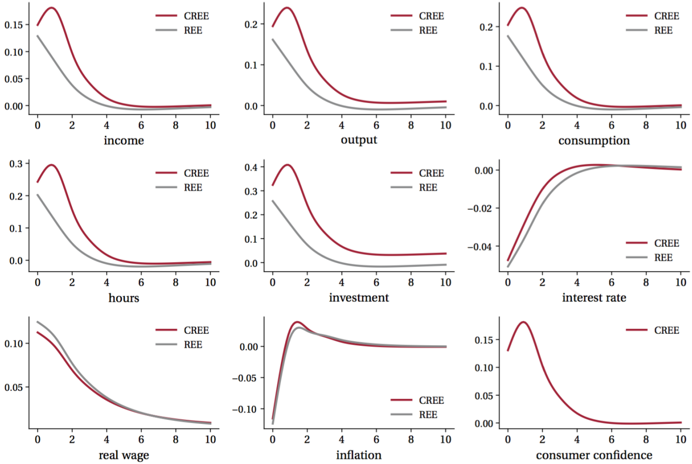

Figure 2. Impulse responses to the TFP shock

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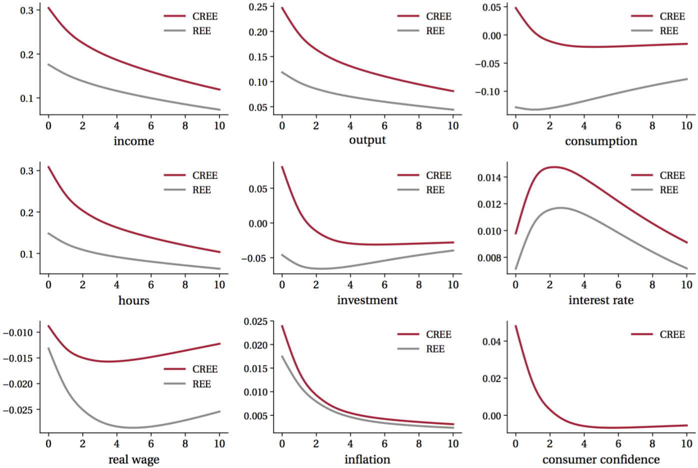

Figure 7. Impulse responses to the government-spending shock

Conclusion

- Good attempt to unify the different models of bounded rationality with a single flexible framework
  - Can be adapted to many macro models
  - Could maybe draw some insights from the stochastic control literature
- Intuitive to think that agents have simpler models of the economy
- Interesting quantitative results