Aggregation, Liquidity, and Asset Prices with Incomplete Markets
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Introduction

- Propose a theory of asset pricing and consumption based in liquidity frictions matching standard facts :
  1. Household consumption not well described by Euler equation,
     High MPC, c.f. Kaplan Violante (2014)
  2. Aggregate consumption follows a simple Euler eq. with the zero-beta rate – cond. expected return on a zero-beta equity portfolio, c.f. Di Tella et (2023))
  3. Aggregate consumption doesn't follow a simple Euler eq. with the safe rate
  4. Security market line is flat : return on zero beta portfolio close to market return.
- This paper :
- Liquidity-based theory of consumption and asset prices
- Aggregation and analytical characterization of asset prices in a two-assets Het. Agent model with idiosyncratic income risk, borrowing constraint and aggregate risk

Werning (2015)

- Aggregation result of HANK models
  - Generalized Euler relation
  $$U'(C_t) = \beta_t R_t U'(C_{t+1})$$
  - Rely on vanishing liquidity (no borrowing and no outside asset) and household income proportional to aggregate income
  - Incomplete market still matters ! change β_t

Assumptions on			Response of aggregate
Income Risk	Liquidity		consumption to interest rates
countercyclical	procyclical	$\rightarrow$	higher sensitivity
acyclical	acyclical	$\rightarrow$	'As if' representative agent
procyclical	countercyclical	$\rightarrow$	lower sensitivity

- Di Tella, Hebert, Kurlat (2024) :
  similar result with Two-Assets HA model matching asset pricing facts.

Log economy

- Household two assets problem :

$$\max_{C_{it},D_{it}} U\left(C_i\right) = \max_{C_{it},D_{it}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \log\left(C_{it}\right) dt\right]$$

$$dA_{it} = \underbrace{D_{it} dN_{it}}_{\substack{\text{deposit} \\ \text{from liquid}}} + \underbrace{r_{at} A_{it} dt + A_{it} \sigma_{at} \cdot dM_t}_{\text{return / agg. risk of illiquid asset}}$$

$$dB_{it} = \Big(\underbrace{e_{it}^0 (1-\alpha) Y_t}_{\substack{\text{labor income} \\ \text{shocks}}} - C_{it}\Big) dt + \underbrace{r_{bt} B_{it} dt + B_{it} \sigma_{bt} \cdot dM_t}_{\text{return / agg. risk of liquid asset}} - \left(D_{it} + \kappa \mathbb{I}_{D_{it} \neq 0} B_t\right) dN_{it}$$

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- Exogenous aggregate income, long-run risk ($\sigma_g > 0$)

$$\frac{dY_t}{Y_t} = g_t dt + \sigma_Y\left(Y_t, g_t\right)\cdot dM_t,$$

$$dg_t = \mu_g\left(g_t, Y_t\right)dt + \sigma_g\left(g_t, Y_t\right)\cdot dM_t$$

Household problem - HJB / KFE

- Defining asset price : $A_t = \mathbb{E}[\int_s e^{-\int_u r_{a,u} du}(1-\theta)\alpha Y_s ds]$ and price-dividend ratio : $P_{at} = A_t / \big((1-\theta)\alpha Y_t\big)$
  - Normalizing : $a_{it} = A_{it}/A_t$, $b_{it} = B_{it}/B_t$, $d_{it} = D_{it}/B_t$, $c_{it} = C_{it}/Y_t$
  - Decision : $c_{it} = (a, b, e; P_{at}, P_{bt}) = c(\cdot)$ depends on the path(s) of M_t through prices.
  - Generator for (a, b, e)

$$\mathcal{L}_{abe}(c_t, d_t; P_{at}, P_{bt}) = \mathcal{L}_e f(\cdot) + \underbrace{\frac{a}{P_{at}}}_{da|dN=0} f_a(\cdot) + \underbrace{\frac{1}{\alpha\theta P_{b,t}}\big(\alpha\theta b + e^0(1-\alpha) - c_t(\cdot)\big)}_{db|dN=0} f_b(\cdot)$$
$$+ \chi\Big(f\big(a + \frac{\theta P_{bt}}{(1-\theta)P_{at}} d_t(\cdot), b - d_t(\cdot) - \kappa \mathbb{I}_{d_t \neq 0}, e\big) - f(a, b, e)\Big)$$

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  - KFE : $d\mu_t(\cdot) = \mathcal{L}^\star(c_t, d_t; P_{at}, P_{bt})\mu_t dt$
    – Note, in theory dM_t should affect μ_t, but not the case because of normalization
  - Steady state : $Y_t{=}1, g_t{=}\sigma_Y{=}\mu_g{=}\sigma_g{=}0,\ \ (r_{at}, P_{at}) = (\bar r_a, \bar P_a)$ and $\bar P_a = 1/\bar r_a$

$$\rho \bar V(a, b, e) = \max_{\substack{c\geq 0 \\ d\in[-a\frac{1-\theta}{\theta}\bar P_a/\bar P_b, b]}} \ln c + \mathcal{L}_{abe}(c, d; \bar P_a, \bar P_b)$$

Markovian equilibrium with aggregate shocks

- Aggregate shocks : Assume $\sigma_Y(Y,g)=0$, Generator for (Y,g)

$$\mathcal{L}_{Yg}f(\cdot)=f_Y(\cdot)gY+f_g(\cdot)\mu_g(Y,g)+\frac{1}{2}f_{gg}(\cdot)\sigma_g(Y,g)^2$$

- ▶ Conjecture equilibrium is Markovian with stationary price/dividend ratio $P_a^\star=\bar{P}_a$
  - Guess and verify : $V(a,b,e;Y,g)=\bar{V}(a,b,e)+\phi(Y,g)$

$$\rho\bar{V}(a,b,e)+\rho\phi(Y,g)=\max_{\substack{c\geq 0\\ d\in[-a\frac{1-\theta}{\theta}\hat{P}_a/\bar{P}_b,b]}}\ln(Y)+\ln c+\mathcal{L}_{abe}\left(c,d;\bar{P}_a,\bar{P}_b\right)\bar{V}(a,b,e)+\mathcal{L}_{Yg}(Y,g)\phi(Y,g)$$

  - $\phi(Y,g)=\mathbb{E}\left[\int_t^\infty e^{-\rho(s-t)}\ln(Y_s)ds\middle|Y_t=Y,g_t=g\right]$
  - ***Proposition 2 :*** If $\sigma_Y=0$, the equilibrium is s.t.

$$\sigma_{at}=\sigma_{bt}=0 \qquad r_{at}=\bar{r}_a+g_t \qquad r_{bt}=\bar{r}_b+g_t \qquad \text{(Euler eq.)}$$
$$c^\star=\bar{c}(\cdot) \qquad d^\star=\bar{d}(\cdot) \qquad \bar{P}_{at}=\bar{P}_a \qquad \bar{P}_{bt}=\bar{P}_b \qquad \mu_t^\star=\bar{\mu}$$

Log economy – Aggregation

- Aggregation results : No redistributive effect of the aggregate shocks
  - Constant shares (L/K, α, Liq./Illiq. θ)
  - Invariance of $\mathcal{L}_e$ to (Y_t, g_t), same as Werning (2015)
- Limitations :
  1. Spread between liquid and illiquid $\bar{r}_a - \bar{r}_b$ constant
     - reflects the convenience of liquid assets for consumption
     - cst because supply of liquid assets $\propto$ liquidity needs from idiosyncratic income shocks e and trading opportunities χ
     - Fact 3 : Euler equation doesn't hold for safe assets
  2. Estimate of Intertemporal Elasticity of Substitution < 1
     - Fact 2 : Euler equation for zero-beta not matched quantitatively
  3. Price dividend ratios $\bar{P}_a, \bar{P}_b$ constant
     - Fact : very volatile procyclical price-dividend ratio

CRRA economy

- Utility $u(C) = \frac{C_{it}^{1-\gamma}}{1-\gamma}$
- Aggregate state of the economy summarized by x_t : claim to agg. conso.
  $\sim$ Price-dividend ratio of RA economy / Wealth-consumption ratio

$$x_t = x(Y_t, g_t) = \rho\Big[\int_t^\infty e^{-\rho(s-t)} \underbrace{\left(\frac{Y_s}{Y_t}\right)^{-\gamma}}_{=SDF} \frac{Y_s}{Y_t} ds \Big| Y_t, g_t\Big]$$

  – Countercyclical x_t : low growth $g_t \Rightarrow$ high x_t

  - Adjust all the generators $\mathcal{L}_{abe}(\cdot, x_t)$ with $\mathcal{L}_e/x_t$ and χ/x_t,
  - Conjecture Markov eq. with $P^\star_{at} = \bar{P}_a/x_t$ + Guess-verify $V(\cdot) = x(Y,g)Y^{1-\gamma}\bar{V}(a,b,e)$
  - ***Proposition 3 :*** If $\sigma_Y = 0$, the equilibrium is s.t.

$$\sigma_{at} = \sigma_{bt} = \frac{\sigma_x(Y_t, g_t)}{x(Y_t, g_t)} \qquad r_{jt} = \rho + \gamma g_t - \frac{\rho - \bar{r}_j}{x_t} \qquad j = a, b \qquad \text{(Euler eq.)}$$

$$c^\star = \bar{c}(\cdot) \qquad d^\star = \bar{d}(\cdot) \qquad \bar{P}_{at} = x_t \bar{P}_a \qquad \bar{P}_{bt} = x_t \bar{P}_b \qquad \mu^\star_t = \bar{\mu}$$

  – Interest r_{jt} falls more than 1-1 with g_t. Spread $s_t = (\bar{r}_a - \bar{r}_b)/x_t$, low g_t high x_t low spread s_t
  Scaling by x_t prevents redistribution from frontloaded income agents to backloaded ones.

Adding aggregate volatility

- Until now $\sigma_Y(\cdot) = 0$, we had $Y_t = C_t$ locally deterministic.
  With $\sigma_Y(\cdot) \neq 0$, can study risk premia + match fact 2.
  - ***Proposition 4 :*** If $\sigma_Y > 0$, the equilibrium is s.t.

$$\sigma_{at} = \sigma_{bt} = \frac{\sigma_x(Y_t, g_t)}{x(Y_t, g_t)} + \sigma_Y(Y_t, g_t) \qquad r_{jt} = \rho + \gamma g_t - \frac{\rho - \bar{r}_j}{x_t} - (\gamma - 1)\frac{\gamma}{2}\sigma_{Yt}^2 + \gamma \frac{\sigma_{xt}}{x_t}\sigma_{Yt} \qquad j = a, b$$

$$c^\star = \bar{c}(\cdot) \qquad d^\star = \bar{d}(\cdot) \qquad \bar{P}_{at} = x_t \bar{P}_a \qquad \bar{P}_{bt} = x_t \bar{P}_b \qquad \mu_t^\star = \bar{\mu}$$

    - Step 1 : Same logic as before + modify generator $\mathcal{L}_{Yg}$
    - Step 2 : Completing markets, add zero-net supply derivative (no trade in equilib.) with return

$$\pi_j(Y, g) = \gamma \sigma_Y(Y_t, g_t) \qquad \text{price of risk}$$

    - Remove the risk from return to get zero-beta $r_{jt}^0 = r_{jt} - \pi_j(Y, g)\left(\frac{\sigma_x(Y_t, g_t)}{x(Y_t, g_t)} + \sigma_Y(Y_t, g_t)\right)$
  - ***Proposition 5*** Consumption CAPM, with zero-beta rates $r_j^0, j = a, b$

$$r_{jt}^0 = \underbrace{\rho + \gamma g_t - (\gamma + 1)\frac{\gamma}{2}\sigma_{Yt}^2}_{=\text{RA Euler eq.}} - \frac{\rho - \bar{r}_j}{x_t}$$

    - Last term : benefit from insurance against idiosyncratic risk ($\neq$ Constantinides, Duffie (1996))

Quantitative evaluation

- How does this model perform vis-a-vis the asset pricing facts ?
  - Works with Price/Divident ratio x_t directly instead of fully specifying the process (Y_t, g_t)

$$\underbrace{\tilde{\rho}}_{=8.5\%} = \rho + \underbrace{\gamma}_{=1/0.2} \underbrace{\mathbb{E}[g_t]}_{=1.5\%} - (\gamma + 1)\frac{\gamma}{2}\sigma_Y^2$$

$$r_{jt}^0 = \mathbb{E}[r_{jt}^0] + \gamma\big(g_t - \mathbb{E}[g_t]\big) - \Big(\tilde{\rho} - \mathbb{E}\big[r_{jt}^0\big]\Big) \times \left(\frac{x_t^{-1}}{\mathbb{E}\big[x_t^{-1}\big]} - 1\right) \qquad (EE)$$

  - Illiquid asset, $\mathbb{E}[r_{at}^0] = 8.5\%$ Perfect fit (fact 2) by construction
  - Liquid asset, $\mathbb{E}[r_{bt}^0] = -1.5\%$. To match fact 3, we need this Euler eq. fails (fact 3)
    - Project x_t^{-1} on growth g_t : $\frac{x_t^{-1}}{\mathbb{E}[x_t^{-1}]} = 1 + \beta\big(g_t - \mathbb{E}_t[g_t]\big) + \epsilon_t, \quad \mathbb{E}[\epsilon_t] = \mathbb{E}[\epsilon_t g_t] = 0$
    - Plug this in (EE) : $r_{bt}^0 - \mathbb{E}[r_{bt}^0] = \big(\gamma - \beta\mathbb{E}[s_t]\big) \times \big(g_t - \mathbb{E}[g_t]\big) - \mathbb{E}[s_t]\epsilon_t$
    - Need ϵ_t to be large ! R^2 of that reg. is 28%, match volatility of dividend/price ratio x_t^{-1}

Asset pricing puzzles

- Equity premium puzzle
  - Calibration match $\mathbb{E}[r^0_{at}]$ and $\mathbb{E}[r^0_{bt}]$
  - To match $r_{at} - r_{bt}$, it requires a small risk premium : $\mathbb{E}[r_{at} - r^0_{at}] = \pi_a(\sigma_x/x + \sigma_Y)$
  - Because liquidity premium already large !
  - $\Rightarrow$ no puzzle, consistent with both CAPM and large equity premium
- Equity volatility puzzle
  - Volatility of illiquid asset $11.3\% = (\sigma_x/x + \sigma_Y) > \sigma_Y$ vol. of consumption growth. the gap comes from dividend/price ratio
- Risk-free rate puzzle
  - With large liquidity premium, easy to match $\mathbb{E}[r^0_{bt}]$ and $std(r_{bt}) = 2.8\%$ (data = 2%)
- Return predictability
  - predictability through valuation ratio x_t^{-1} : $r^0_{at} - r^0_{bt} = \left(\mathbb{E}[r^0_{at}] - \mathbb{E}[r^0_{bt}]\right)\frac{x_t^{-1}}{\mathbb{E}[x_t^{-1}]}$