Mathematical Methods in Economics
Lecture Notes

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Contents

6 Control theory in discrete time **2**
6.1 Sequential approach . . . 2
6.2 Recursive approach – Dynamic programming . . . 10

7 Control theory in continuous time **27**
7.1 Sequential approach and Pontryagin Maximum principle . . . 27
7.2 Recursive approach and Hamilton-Jacobi-Bellman equations . . . 33

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6 Control theory in discrete time

Optimal control stems from the older field of optimization and calculus of variation (optimization in infinite dimension) and look for optimal paths of dynamical systems, possibly subject to exogenous stochastic disturbance. The two main contributors to this field have been L. Pontryagin (USSR) in Russia and R. Bellman in the U.S. in the 1950s. In its simplest form, it related to an extension of Kuhn-Tucker theorem of optimization.

6.1 Sequential approach

In the following we consider problem expressed as a control-state problem : a state move dynamically and the optimizers choose the control process that has an influence in the dynamics.

A typical optimal control problem is as follows:

$$\begin{aligned} V(x_0) = \max_{\{c_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t F(x_t, c_t) && (P_{c-s}) \\ \text{s.t. } \; & x_{t+1} = g(x_t, c_t) \\ & c_t \in \Gamma(x_t) \\ & x_0 \text{ given} && (1) \end{aligned}$$

where

- $\{x_t\}_t$ is the state variable process, included in a space $\mathbb{X}$
- $\{c_t\}$ a control/choice variables process in the space $\mathbb{C}$.
- $\Gamma(x_t)$ gives the set of feasible values given the states, represented by a correspondence (i.e. multi-valued function) $\Gamma : \mathbb{X} \mapsto \mathbb{C}$.
- $F : \mathbb{X} \times \mathbb{C} \to \mathbb{R}$ is the period utility/profit or "running gain" from holding a state x_t and choosing the control c_t.
- $\beta \in (0, 1)$ is the discount factor.

There are different approaches to this setting. The first one is to keep the control-state formulation for (P_{c-s}) and derive the optimality conditions using the Lagrangian of the problem (in many dimensions). The other one, more suited for the proofs, express the problem with only states variables (presents and futures).

Control-state formulation

The Lagrangian associated with the problem is:

$$\mathcal{L}\big(\{c_t, x_{t+1}, \lambda_t\}_{t\geq 0}\big) = \sum_{t=0}^{\infty} \beta^t \Big(F(x_t, c_t) - \lambda_t \big(x_{t+1} - g(x_t, c_t)\big)\Big)$$

Proposition 6.1 (Necessary and sufficient conditions – Control-States formulation)**.**
Let $\mathbb{X}\times\mathbb{C} \subset \mathbb{R}^N \times \mathbb{R}^K$*, if* $\{x_{t+1}, c_t\}_{t=0}^{\infty}$ *is the solution of the problem* (P_{c-s})*, then*

1. $\frac{\partial}{\partial c}F(x_t, c_t) + \lambda_t \frac{\partial}{\partial c} g(x_t, c_t) = 0$
2. $\lambda_t = \beta\Big(\frac{\partial}{\partial x}F(x_{t+1}, c_{t+1}) + \lambda_{t+1}\frac{\partial}{\partial x} g(x_{t+1}, c_{t+1})\Big)$
3. $x_{t+1} = g(x_t, c_t)$

These necessary conditions are also sufficient:
Suppose that the necessary conditions are associated with the following assumptions

1. $\widehat{F}(x_t, x_{t+1}) \coloneqq F(x_t, c(x_t, x_{t+1}))$ *is concave in* x_t, x_{t+1}
2. $\frac{\partial}{\partial y}\widehat{F}(x_t, x_{t+1}) \leq 0$ *where* y *denotes the second variable.*
3. $\lim_{t\to\infty} \beta^t \lambda_t x_{t+1} = 0$

then all these conditions are sufficient for optimality of the path $\{x_t, c_t\}_t$
Note: The proof of this theorem can be found in the book RMED by Lucas and Stokey, p 98.

Let us develop the most standard example that we covered in class.

Example 6.1 (Neoclassical growth model: from finite to infinite horizon)**.**
Before going to the infinite horizon example, we first consider a finite horizon to build intuitions and then take the finite time T *to the limit* $T \to \infty$*.*

Consider a planner who wish to maximize the discounted sum of period utility $u(c_t)$ *with discount rate* β*. Assume that* $u' > 0, u'' < 0, \lim_{c\to 0} u' = \infty$*.*

The planner is endowed with initial capital k_0*. With* k_t *amount of capital, the planner receives* $f(k_t) + (1-\delta)k_t$ *units of goods, which can be use for consumption* c_t *and accumulation of new capital* k_{t+1}*. Assume that* $f(0) = 0, f' > 0, f'' < 0$*.*

The problem of the planner has the following formulation:

$$\begin{aligned} &\max_{\{c_t, k_{t+1}\}_{t\geq 0}} \sum_{t=0}^{T} \beta^t u(c_t) \\ &\quad s.t.\ k_{t+1} = f(k_t) + (1-\delta)k_t - c_t \qquad \forall 0 \leq t \leq T \\ &\qquad\quad k_t \geq 0, c_t \geq 0, \qquad \forall 0 \leq t \leq T+1 \\ &\qquad\quad k_0 \text{ given} \end{aligned}$$

With the assumption on $u(\cdot)$ *and* $f(\cdot)$ *above, it is easy to see that the constraints* $k_t \geq 0, c_t \geq 0$ *are never binding on an optimal path for* $t \leq T$*. The Lagrangian of the problem is then given by:*

$$\mathcal{L}(\{c_t, k_{t+1}, \lambda_t\}_t) = \sum_{t=0}^{T} \beta^t \Big(u(c_t) + \lambda_t(f(k_t) + (1-\delta)k_t - c_t - k_{t+1})\Big) + \eta\ k_{T+1}$$

where $\{\lambda_t\}, \eta$ *are the multipliers.*

Since there are 2 controls variables per period c_t, k_{t+1} *plus a final positivity constraint for capital* $k_{T+1} \geq 0$ *in the last period, we obtain* $2 \times t + 1$ *F.O.C. (First order conditions). This gives:*

$$
\begin{aligned}
&[c_t] : u'(c_t) - \lambda_t = 0 \\
&[k_{t+1}] : -\lambda_t + \beta\lambda_{t+1}(f'(k_{t+1}) + (1-\delta)) = 0, \forall t < T \\
&[k_{T+1}] : -\beta^T \lambda_T + \eta = 0
\end{aligned}
$$

and we have complementary slackness

$$\eta \; k_{T+1} = 0.$$

Rearranging the F.O.C.'s gives the Euler equation:

$$u'(c_t) = \beta u'(c_{t+1})\big(f'(k_{t+1}) + 1 - \delta\big).$$

Notice that the Euler equation together with the law of motion for capital consists a system of difference equation of $\{c_t, k_t\}_t$*:*

$$
\begin{aligned}
u'(c_t) &= \beta u'(c_{t+1})\big(f'(k_{t+1}) + 1 - \delta\big) \\
k_{t+1} &= f(k_t) + (1-\delta)k_t - c_t
\end{aligned}
$$

With initial condition k_0 *given, fix any feasible* c_0*, we can solve the system forward for* (c_1, k_1)*,* (c_2, k_2) *and so on. Hence, there are infinitely many sequences (depending on the* c_0 *choice) that satisfies the Euler equation and the law of motion.*

How can we pin down the optimal sequence then? We have one more equation:

$$0 = \eta k_{T+1} = \beta^T \lambda_T k_{T+1} = \beta^T u'(c_T) k_{T+1}$$

This condition says that if the date T *marginal value of consumption* $u'(c_T)$ *is positive (which is true in our case), then the optimal* k_{T+1} *must be zero. The reason is simple: the agent would have consumed a little bit more at date* T *and leave less capital behind if* $k_{T+1} > 0$*. Said differently, if the marginal value of income* λ_T *(which is also the marginal utility of consumption* $u'(c_T)$*) is strictly positive, then there are incentive the reduce future wealth down* $k_{T+1} = 0$*.*

Therefore, besides the initial value k_0*, optimality gives us another bounder condition at date* T*:*

$$k_T = 0.$$

This uniquely pins down the optimal sequence $\{c_t, k_t\}$*.*

Now, for the infinite horizon problem*, one needs to redo the same steps as above. The last complementary slackness becomes the transversality condition:*

$$\lim_{t\to\infty} \beta^t \lambda_t k_{t+1} = 0.$$

and together with Euler equation and the law of motion for capital is sufficient for optimality.

The transversality condition plays the role of a boundary condition at $t \to \infty$ *as the complementary slackness condition* $\lambda_T \ k_{T+1} = 0$ *did in the finite horizon case.*

. . . ——— . . .

State formulation

The second approach, to be able to prove necessary and sufficient conditions for optimality, one needs to reformulate the problem with "states" mostly to be able to apply KKT directly[1].

$$V(x_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \widehat{F}(x_t, x_{t+1}) \tag{P_s}$$
$$\text{s.t. } x_{t+1} \in \widehat{\Gamma}(x_t)$$
$$x_0 \text{ given}$$

$$\widehat{F}(x,y) = \max_c \{F(x,c) : c \in \Gamma(x), y = g(x,c)\} \tag{2}$$

$$\widehat{\Gamma}(x) = \{y \ s.t. \exists c \in \Gamma(x) \ \& \ y = g(x,c)\} \tag{3}$$

with $\widehat{\Gamma}(x_t)$ gives the set of feasible values of future states, again represented by a correspondence (i.e. multi-valued function) $\Gamma : \mathbb{X} \mapsto \mathbb{X}$, and $\widehat{F} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ is the period utility/profit or "running gain" from holding a state x_t and choosing the future state x_{t+1}.

Definition 6.1 (Euler Equations – discrete time)**.**
Assume that $\mathbb{X} \in \mathbb{R}^n$*, and the function*[2] F *is* C^1*. The path* $\{x_{t+1}\}_{t=0}^{\infty}$ *satisfies EE if*

$$F_y(x_t, x_{t+1}) + \beta F_x(x_{t+1}, x_{t+2}) = 0, \forall t \geq 0$$

If x *is a vector, then* EE *must hold for each dimension of* x*. the Definition* 7.2.

Definition 6.2 (Transversality Condition – discrete time)**.**
The path $\{x_{t+1}\}_{t=0}^{\infty}$ *satisfies TC if*

$$\lim_{t\to\infty} \beta^t F_x(x_t, x_{t+1}) \cdot x_t = 0$$

[1] And also because that is the way F. Alvarez will introduce it during the year in his course Theory of Income

[2] Here and in the following we redefine the function $\widehat{F} \equiv F$

Theorem 6.2 (Sufficiency)**.**
We now show that (EE) *and* (TC) *are sufficient for optimality if the problem is convex. Assume that $\widehat{F}$ is concave in $(x, y), F_x\left(x_t^*, x_{t+1}^*\right) \geq 0$ and $\mathbb{X} = \mathbb{R}_+^n$ Then, if $\left\{x_{t+1}^*\right\}_{t=0}^{\infty}$ satisfies EE and TC, the path $\left\{x_{t+1}^*\right\}_{t=0}^{\infty}$ is optimal.* Test

Proof

We use the fact that, if f is concave the tangency line at any given point always lies above f; i.e.

$$f(z) \leq f\left(z^0\right) + f'\left(z^0\right)\left(z - z^0\right), \ \forall z. \tag{4}$$

Now, take an arbitrary path that has the same initial condition as the optimal path: i.e. take $\{x_{t+1}\}_{t=0}^{\infty}$ with $x_0 = x_0^*$. By the assumption that $\mathbb{X} = \mathbb{R}_+^n$, $x_{t+1} \geq 0$ for all t. We wish to show that $F\left(x_t^*, x_{t+1}^*\right)$ is greater than $F\left(x_t, x_{t+1}\right)$ across all periods; i.e. we want to show that:

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[F\left(x_t, x_{t+1}\right) - F\left(x_t^*, x_{t+1}^*\right)\right] \leq 0.$$

Note that we can rearrange $f(z) - f\left(z^0\right) \leq f'\left(z^0\right)\left(z - z^0\right)$, and by letting $z^0 = x^*$,

$$F\left(x_t, x_{t+1}\right) - F\left(x_t^*, x_{t+1}^*\right) \leq F_x\left(x_t^*, x_{t+1}^*\right)\left(x_t - x_t^*\right) + F_y\left(x_t^*, x_{t+1}^*\right)\left(x_{t+1} - x_{t+1}^*\right)$$

That is,

$$\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[F\left(x_t, x_{t+1}\right) - F\left(x_t^*, x_{t+1}^*\right)\right] \leq \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[F_x\left(x_t^*, x_{t+1}^*\right)\left(x_t - x_t^*\right) + F_y\left(x_t^*, x_{t+1}^*\right)\left(x_{t+1} - x_{t+1}^*\right)\right]$$

Expanding the summation, we have that

$$\begin{aligned}
&\sum_{t=0}^{T} \beta^t \left[F_x\left(x_t^*, x_{t+1}^*\right)\left(x_t - x_t^*\right) + F_y\left(x_t^*, x_{t+1}^*\right)\left(x_{t+1} - x_{t+1}^*\right)\right] \\
&= F_x\left(x_0^*, x_1^*\right)\left(x_0 - x_0^*\right) + F_y\left(x_0^*, x_1^*\right)\left(x_1 - x_1^*\right) \\
&\quad + \beta\left(F_x\left(x_1^*, x_2^*\right)\left(x_1 - x_1^*\right) + F_y\left(x_1^*, x_2^*\right)\left(x_2 - x_2^*\right)\right) \\
&\quad + \cdots \\
&\quad + \beta^t\left(F_x\left(x_t^*, x_{t+1}^*\right)\left(x_t - x_t^*\right) + F_y\left(x_t^*, x_{t+1}^*\right)\left(x_{t+1} - x_{t+1}^*\right)\right) \\
&\quad + \cdots \\
&\quad + \beta^T\left(F_x\left(x_T^*, x_{T+1}^*\right)\left(x_T - x_T^*\right) + F_y\left(x_T^*, x_{T+1}^*\right)\left(x_{T+1} - x_{T+1}^*\right)\right)
\end{aligned}$$

By assumption, $x_0 = x_0^*$ so that the first term is zero. Rewriting the expression gives

$$\begin{aligned}
&\sum_{t=0}^{T} \beta^t \left[F_x\left(x_t^*, x_{t+1}^*\right)\left(x_t - x_t^*\right) + F_y\left(x_t^*, x_{t+1}^*\right)\left(x_{t+1} - x_{t+1}^*\right)\right] \\
&= \left[F_y\left(x_0^*, x_1^*\right)\left(x_1 - x_1^*\right) + \beta\left(F_x\left(x_1^*, x_2^*\right)\left(x_1 - x_1^*\right)\right)\right] \\
&\quad + \beta\left[F_y\left(x_1^*, x_2^*\right)\left(x_2 - x_2^*\right) + \beta F_x\left(x_2^*, x_3^*\right)\left(x_2 - x_2^*\right)\right] \\
&\quad + \cdots \\
&\quad + \beta^t\left[F_y\left(x_t^*, x_{t+1}^*\right)\left(x_{t+1} - x_{t+1}^*\right) + \beta F_x\left(x_{t+1}^*, x_{t+2}^*\right)\left(x_{t+1} - x_{t+1}^*\right)\right] \\
&\quad + \cdots \\
&\quad + \beta^T F_y\left(x_T^*, x_{T+1}^*\right)\left(x_{T+1} - x_{T+1}^*\right)
\end{aligned}$$

where we realize that the terms inside the square brackets are all zero by the EE. Thus, the expression above simplifies to

$$\sum_{t=0}^{T}\beta^t\left[F_x\left(x_t^*,x_{t+1}^*\right)\left(x_t-x_t^*\right)+F_y\left(x_t^*,x_{t+1}^*\right)\left(x_{t+1}-x_{t+1}^*\right)\right]=\beta^TF_y\left(x_T^*,x_{T+1}^*\right)\left(x_{T+1}-x_{T+1}^*\right)$$

From EE, we know that $F_y\left(x_t,x_{t+1}\right)=-\beta F_x\left(x_{t+1},x_{t+2}\right)$, hence we can write

$$\beta^TF_y\left(x_T^*,x_{T+1}^*\right)\left(x_{T+1}-x_{T+1}^*\right)=-\beta^{T+1}F_x\left(x_{T+1}^*,x_{T+2}^*\right)\left(x_{T+1}-x_{T+1}^*\right)$$

By assumption, we know that $x_{T+1}\geq 0$ and $F_x\left(x_{T+1}^*,x_{T+2}^*\right)\geq 0$ so that

$$\begin{aligned}\beta^TF_y\left(x_T^*,x_{T+1}^*\right)\left(x_{T+1}-x_{T+1}^*\right)&=-\beta^{T+1}F_x\left(x_{T+1}^*,x_{T+2}^*\right)\left(x_{T+1}-x_{T+1}^*\right)\\&=\underbrace{-\beta^{T+1}F_x\left(x_{T+1}^*,x_{T+2}^*\right)x_{T+1}}_{\leq 0}+\beta^{T+1}F_x\left(x_{T+1}^*,x_{T+2}^*\right)x_{T+1}^*\\&\leq\beta^{T+1}F_x\left(x_{T+1}^*,x_{T+2}^*\right)x_{T+1}^*.\end{aligned}$$

By assumption, TC holds so that

$$\lim_{T\to\infty}\sum_{t=0}^{T}\beta^t\left[F\left(x_t,x_{t+1}\right)-F\left(x_t^*,x_{t+1}^*\right)\right]\leq\lim_{T\to\infty}\beta^{T+1}F_x\left(x_{T+1}^*,x_{T+2}^*\right)x_{T+1}^*=0.$$

□

Note that this proof is in essence the same as the sufficiency condition proof of the Kuhn Tucker theorem. We proved sufficient (thanks to convexity of the problem), and we now prove that EE and TC are also necessary conditions for the path $\{x_t\}_t$ to be optimal.

Proposition 6.3 (Necessity).
Assume that F is C^1 and $\left\{x_{t+1}^*\right\}_{t=0}^{\infty}$ is optimal. Then, $\left\{x_{t+1}^*\right\}_{t=0}^{\infty}$ satisfies EE and TC.

Proof

We will consider adding perturbations to the optimal path $\{x_t^*\}_{t=1}^{\infty}$, denoted by ε. Let

$$x_t\left(\alpha,\varepsilon\right)=x_t^*+\alpha\varepsilon_t,\ \forall t\geq 0$$

for $\alpha\in\mathbb{R}$ and $\varepsilon=\{\varepsilon_t\}_{t=0}^{\infty}$ with $\varepsilon_t\in\mathbb{R}^n$ and $\varepsilon_0=0$ (again, we must start from the same point). Since $\{x_{t+1}^*\}_{t=0}^{\infty}$ is optimal, it must be that

$$V^*\left(x_0\right)=v\left(0\right):=\lim_{T\to\infty}\sum_{t=0}^{T}\beta^tF\left(x_t\left(0,\varepsilon\right),x_{t+1}\left(0,\varepsilon\right)\right)\geq v\left(\alpha\right):=\lim_{T\to\infty}\sum_{t=0}^{T}\beta^tF\left(x_t\left(\alpha,\varepsilon\right),x_{t+1}\left(\alpha,\varepsilon\right)\right)$$

for any α,ε such that $x_{t+1}\left(\alpha,\varepsilon\right)\in\Gamma\left(x_t\left(\alpha,\varepsilon\right)\right),\ \forall t\geq 0$ (i.e. perturbed path must still be feasible).

Since $\alpha=0$ maximises v, if v is differentiable, it must be that

$$\frac{\partial v\left(0\right)}{\partial\alpha}=0.$$

Assuming that the limits involved in the derivative (with respect to α) and in the summation (with respect to T) can be exchanged, we obtain that (since $\partial x_t\left(\alpha,\varepsilon\right)/\partial\alpha=\varepsilon_t$),

$$\frac{\partial v\left(0\right)}{\partial\alpha}=\lim_{T\to\infty}\sum_{t=0}^{T}\beta^t\left[F_x\left(x_t^*,x_{t+1}^*\right)\varepsilon_t+F_y\left(x_t^*,x_{t+1}^*\right)\varepsilon_{t+1}\right].$$

Consider the summation separately,

$$\begin{aligned}
&\sum_{t=0}^{T} \beta^t \left[F_x\left(x_t^*, x_{t+1}^*\right)\varepsilon_t + F_y\left(x_t^*, x_{t+1}^*\right)\varepsilon_{t+1}\right] \\
&= F_x\left(x_0^*, x_1^*\right)\varepsilon_0 + F_y\left(x_0^*, x_1^*\right)\varepsilon_1 \\
&\qquad + \beta\left[F_x\left(x_1^*, x_2^*\right)\varepsilon_1 + F_y\left(x_1^*, x_2^*\right)\varepsilon_2\right] \\
&\qquad + \cdots \\
&\qquad + \beta^t\left[F_x\left(x_t^*, x_{t+1}^*\right)\varepsilon_t + F_y\left(x_t^*, x_{t+1}^*\right)\varepsilon_{t+1}\right] \\
&\qquad + \cdots \\
&\qquad + \beta^T\left[F_x\left(x_T^*, x_{T+1}^*\right)\varepsilon_T + F_y\left(x_T^*, x_{T+1}^*\right)\varepsilon_{T+1}\right].
\end{aligned}$$

By the assumption that $\varepsilon_0 = 0$, the first term is zero. Then, we can write

$$\begin{aligned}
\frac{\partial v(0)}{\partial \alpha} &= \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[F_x\left(x_t^*, x_{t+1}^*\right)\varepsilon_t + F_y\left(x_t^*, x_{t+1}^*\right)\varepsilon_{t+1}\right] \\
&= \lim_{T\to\infty} \sum_{t=0}^{T-1} \beta^t \left(F_y\left(x_t^*, x_{t+1}^*\right) + \beta F_x\left(x_{t+1}^*, x_{t+2}^*\right)\right)\varepsilon_{t+1} + \lim_{T\to\infty} \beta^T F_y\left(x_T^*, x_{T+1}^*\right)\varepsilon_{T+1}.
\end{aligned}$$

Consider the case where $\varepsilon_s = 0$ for all s except at time $t+1$. In this case, $x_{t+1}(\alpha, \varepsilon)$ will be feasible if $\left(x_{t+1}^*, x_t^*\right) \in \operatorname{int}\left(\operatorname{Gr}(\Gamma)\right)$ (if it was on the boundary, then perturbing would lead to $x_{t+1}(\alpha, \varepsilon)$ that is not feasible). Also, assume that v is differentiable and the limits can be interchanged. Then, it must be that

$$\frac{\partial v(0)}{\partial \alpha} = \left[F_y\left(x_t^*, x_{t+1}^*\right) + \beta F_x\left(x_{t+1}^*, x_{t+2}^*\right)\right]\varepsilon_{t+1} = 0.$$

Since this must hold for any ε_{t+1} in the neighbourhood of 0 such that $x_{t+1}(\alpha, \varepsilon)$ is feasible, it must be that the term inside the square brackets sum to zero; i.e.

$$F_y\left(x_t^*, x_{t+1}^*\right) + \beta F_x\left(x_{t+1}^*, x_{t+2}^*\right) = 0.$$

Since this must be true for any $t+1$, we obtain the EE and that

$$0 = \frac{\partial v(0)}{\partial \alpha} = \lim_{T\to\infty} \beta^T F_y\left(x_T^*, x_{T+1}^*\right)\varepsilon_{T+1}.$$

Using the EE that we've already established, we know $F_y\left(x_T^*, x_{T+1}^*\right) = -\beta F_x\left(x_{T+1}^*, x_{T+2}^*\right)$ so that

$$0 = \frac{\partial v(0)}{\partial \alpha} = -\lim_{T\to\infty} \beta^{T+1} F_x\left(x_{T+1}^*, x_{T+2}^*\right)\varepsilon_{T+1}.$$

Finally, if $\varepsilon_{T+1} = -x_{T+1}^*$ is feasible then

$$0 = \frac{\partial v(0)}{\partial \alpha} = \lim_{T\to\infty} \beta^T F_x\left(x_T^*, x_{T+1}^*\right)x_T^*.$$

That is, TC must hold.

□

Sequential problem with stochastic dynamics

We can assume that the problem is subject to exogenous stochastic shocks where $\{\varepsilon_t\}$ have an influence on the dynamics of the states. The agents or social planner will take the value of state and the information available at time 0 (the σ-algebra given by the random variable ε_0). Because of this introduction of risk (under the form of an exogenous disturbance w_t), all the variables become stochastic processes. More precisely $\varepsilon_t, x_{t+1}, c_t$ and λ_t are stochastic processes measurable with respect to $\mathscr{F}_t = \sigma(\varepsilon_t)$ the information-set at time t (i.e. the σ-algebra generated by the random variable ε_t).

$$\max_{\{x_t, c_t\}_{t=0}^{\infty}} \mathbb{E}\Big[\sum_{t=0}^{\infty} \beta^t F(x_t, c_t, \varepsilon_t) \Big| \sigma(\varepsilon_0)\Big] \tag{P_{c-s-r}}$$

$$\begin{aligned} \text{s.t. } & x_{t+1} = g(x_t, c_t, \varepsilon_t) \\ & \varepsilon_{t+1} = h(\varepsilon_t, w_t) \\ & x_0, \varepsilon_0 \text{ given} \end{aligned}$$

The Lagrangian function of the problem is :

$$\mathcal{L}(\{c_t, x_{t+1}, \lambda_t\}_{t\geq 0}) = \mathbb{E}_0\Big[\sum_{t=0}^{\infty} \beta^t \big(F(x_t, c_t, \varepsilon_t) + \lambda_t(g(x_t, c_t, \varepsilon_t) - x_{t+1})\big)\Big]$$

The control-state formulation and the necessary and sufficient conditions are the following.

Proposition 6.4 (Necessary and sufficient conditions – Control-State formulation with risk)**.** *Let $\mathbb{X}\times\mathbb{C} \subset \mathbb{R}^N \times \mathbb{R}^K$, if $\{x_{t+1}, c_t\}_{t=0}^{\infty}$ is the solution of the above problem, then*

1. $\frac{\partial}{\partial c} F(x_t, c_t, \varepsilon_t) + \lambda_t \frac{\partial}{\partial c} g(x_t, c_t, \varepsilon_t) = 0$
2. $\lambda_t = \beta \mathbb{E}_t\Big(\frac{\partial}{\partial x} F(x_{t+1}, c_{t+1}, \varepsilon_{t+1}) + \lambda_{t+1} \frac{\partial}{\partial x} g(x_{t+1}, c_{t+1}, \varepsilon_{t+1})\Big)$
3. $x_{t+1} = g(x_t, c_t, \varepsilon_t)$ *and* $\varepsilon_{t+1} = h(\varepsilon_t, w_t)$

where $\mathbb{E}_t[.] = \mathbb{E}[.|\sigma(\varepsilon_t)]$. These necessary conditions are also sufficient:
Suppose that the necessary conditions are associated with the following assumptions

1. $\widehat{F}(x_t, x_{t+1}) \coloneqq F(x_t, c(x_t, x_{t+1}))$ *is concave in* x_t, x_{t+1}
2. $\frac{\partial}{\partial y} \widehat{F}(x_t, x_{t+1}) \leq 0$ *where y denotes the second variable.*
3. $\lim_{t\to\infty} \mathbb{E}_0\big(\beta^t \lambda_t x_{t+1}\big) = 0$

then all these conditions are sufficient for optimality of the path $\{x_t, c_t\}_t$

6.2 Recursive approach – Dynamic programming

In the recursive approach, in the difference of the sequential approach the optimizer (social planner or economic agents) do not optimize w.r.t to the entire 'sequence'/path of controls and states. In the contrary would choose its decision at each stage. The main idea of R. Bellman was that he could recompose the path of optimality in difference "stages", where each stage is a simpler problem (in smaller dimension), and we can use the recursive structure to solve the problem in a local way.

Let us consider the same problem (P_{c-s}) as above;

$$\begin{aligned} \max_{\{x_{t+1},c_t\}} \ & \sum_{t=0}^{\infty} \beta^t F(x_t, c_t) \\ \text{s.t. } \ & x_{t+1} = g(x_t, c_t) \\ & c_t \in \Gamma(x_t) \\ & x_0 \text{ given} \end{aligned}$$

where the definitions of the object are the same as eq. (P_{c-s}). Note that again,

Definition 6.3 (Feasibility)**.**
The couple of sequence $\{x_t, c_t\}_{t\geq 0}$ *is a feasible plan given* x_0 *if*

$$\begin{aligned} c_t \in & \Gamma(x_t) \\ x_{t+1} = & g(x_t, c_t) \ \forall t, \ x_0 \ \textit{given} \end{aligned}$$

we denote $\Pi(x_0)$ *the set of feasible plan given* x_0*.*

Note that if $\{x_t, c_t\}_{t\geq 0\}}$ is a feasible plan given x_0, then $\{x_{t+\tau}, c_{t+\tau}\}_{t\geq 0}$ is a feasible plan given x_τ for $\tau \geq 0$.

Theorem 6.5 (Value function and Bellman equation)**.**
Define the value function as :

$$\begin{aligned} J(x_{t+1}, \{c_t\}_{t\geq 0}) := & \sum_{t=0}^{\infty} \beta^t F(x_t, c_t) \\ V(x_0) = & \sup_{\{c_t\}_{t\geq 0} \in \Pi(x_0)} J(\{c_t\}_{t\geq 0}), \end{aligned}$$

We can write the value function $V(x_0)$ *recursively as*

$$V(x_0) = \max_{c_0 \in \Gamma(x_0)} F(x_0, c_0) + \beta V(g(x_0, c_0)) \tag{$P_{B.e.}$}$$

Proof

Let us derive a construction for the recursive form $\widehat{V}(x_t)$:

$$
\begin{aligned}
V(x_t) &= \sup_{\{c_{t+k}\}_{k\geq 0}\in\Pi(x_t)} J(x_{t+1}, \{c_{t+k}\}_{k\geq 0}) \\
&= \sup_{\{c_{t+k}\}_{k\geq 0}\in\Pi(x_t)} \sum_{k=0}^{\infty} \beta^k F(x_{t+k}, c_{t+k}) \\
&= \sup_{\substack{c_t\in\Gamma(x_t),\\ \{c_{t+1+k}\}_{k\geq 0}\in\Pi\left(g(x_t,c_t)\right)}} F(x_t, c_t) + \sum_{k=1}^{\infty} \beta^k F(x_{t+k}, c_{t+k}) \\
&= \sup_{c_t\in\Gamma(x_t)} F(x_t, c_t) + \sup_{\{c_{t+1+k}\}_{k\geq 0}\in\Pi\left(g(x_t,c_t)\right)} \sum_{k'=0}^{\infty} \beta^{k'+1} F(x_{t+1+k'}, c_{t+1+k'}) \\
&= \sup_{c_t\in\Gamma(x_t)} F(x_t, c_t) + \beta \sup_{\{c_{t+1+k}\}_{k\geq 0}\in\Pi\left(g(x_t,c_t)\right)} J\big(g(x_t, c_t), \{c_{t+1+k}\}_{k\geq 0}\big) \\
&= \sup_{c_t\in\Gamma(x_t)} F(x_t, c_t) + \beta V(g(x_t, c_t))
\end{aligned}
$$

Or more simply with "dummy variables" x and c and assuming that suprema are reached (more on that below).

$$
V(x) = \max_{c\in\Gamma(x)} F(x, c) + \beta V(g(x, c))
$$

Let us prove necessity and sufficiency of this recursive formulation.

Given x_0, suppose an optimal exists and is given by $\{c_t^*\}_{t\geq 0}$. Denote $\{x_t^*\}_t$ the controlled dynamics given by the function $g(\cdot)$ and $\{c_t^*\}_{t\geq 0}$. then

$$
V(x_0) = \sum_{t=0}^{\infty} \beta^t F(x_t^*, u_t^*).
$$

And let the recursive formulation denoted by $\widehat{V}(x_0)$ as:

$$
\widehat{V}(x_0) \coloneqq \max_{c_0\in\Gamma(x_0)} F(x_0, u_0) + \beta V(g(x_0, c_0)).
$$

Then

$$
V(x_0) = F(x_0, c_0^*) + \beta \sum_{t=0}^{\infty} \beta^t F(x_{t+1}^*, c_{t+1}^*) = F(x_0^*, c_0^*) + \beta J(x_0, \{c_{t+1}^*\}) \leq F(u_0^*) + \beta V\big(g(x_0, c_0^*\big) \leq \hat{V}(x_0)
$$

the first inequality follows since $(g(x_0, c_0^*), \{c_{t+1}^*\}_t)$ is feasible and the second from the definition of $\widehat{V}(x_0)$.

Conversely, let $\hat{c}_0$ be the optimizer for $\hat{V}(x_0)$ and $\{c'_{t+1}\}$ is the optimal plan that attain the value $V\big(g(x_0, \hat{c}_0)\big)$. Then

$$
V(x_0) \geq F(x_0, \hat{c}_0) + \beta \sum_{t=0}^{\infty} \beta^t F(x_{t+1}, c'_{t+1}) = F(x_0, \hat{c}_0) + \beta V(g(x_0, \hat{c}_0) = \hat{V}(x_0)
$$

since $\big\{g(x_0, c_0^*), \{c'_{t+1}\}_t\big)$ is feasible implies $\big(x_0, \{c_0^*, c'_1, \dots\}_t\big)$ is feasible.

Hence, $V(x_0) = \hat{V}(x_0)$ and it satisfies the Bellman equation.

On the other hand, starting from the Bellman equation, if F is bounded under all feasible plans, then the solution of the functional equation is the value function.

□

Working with Bellman equations – A cookbook

Let us derive fully an example as we did in class:

Example 6.2 (Neoclassical model with labor and capital)**.**
Now, let us consider the Household problem:

$$\max_{\{c_t,k_{t+1},\ell_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t U(c_t, 1-\ell_t)$$
$$s.t. \qquad c_t + k_{t+1} \leq F(k_t, \ell_t)$$

where c, k' and ℓ and $1-\ell$ are respectively consumption, future capital, labor and leisure. The problem can be reformulated recursively with a "Bellman" equation:

$$V(k) = \max_{c,k',\ell} \Big[U(c, 1-\ell) + \beta V(k')\Big]$$

To solve this problem, you need to derive (i) the KKT FOC-conditions for this new (iterative) maximization problem[3]*, and (ii) an "Envelope condition" showing the derivative of the value function (w.r.t. k). Combining the different equations and manipulating the terms, we should obtain: (a) the Euler equation and (b) the Labor-consumption trade-off.*

(0.) As a preliminary exercise, let us recall the *Envelope Theorem*. This theorem arises when one want to differentiate a function that is expressed as an optimization problem. The most important case in economics (micro and macro) is when a value function depends on a "state-variable" x but is the optimum over a "control" variable y:

$$v(x) = \max_{y \in K} f(x, y) \qquad \forall x \in \mathbb{R}^d$$

We develop the multivariate case here, where: K is a compact set (eventually the set resulting from a set of constraints) $K \subset \mathbb{R}^m$ where m is the dimension of the control (y) and d the dimension of the state (x). We assume that, for every value of $y \in K$, $f(.,y)$ is differentiable, and $\nabla_x f(x,y)$ is continuous.

We assume, after using the KKT theorem above if the case is appropriate, that this maximization problem has an unique optimum. This optimal y^* depends on the state variable: $y^*(x)$, implicitly derived from the KKT optimality conditions (and that can be analytically expressed using the ... implicit theorem!). In macro, the result of the maximization problem is often called the *policy function*: $y^*(x) = \pi(x)$.

Given all these assumptions, the value function v is of class C^1 and we can explicitly obtain its derivative:

$$\nabla v(x) = \nabla_x f(x, y^*(x)), \qquad \forall x \in \mathbb{R}^d$$

[3] Note that we are back to the first "static" version of KKT theorem

Important remark: the main thing here is that the derivative v is the same as the derivative of f w.r.t. its first variable, and $\nabla_y f(x, y^*(x)) = 0$. Why? Because the function $f(., y)$ is already optimal w.r.t. y and implied by the (KKT-) FOC conditions. Therefore, the chain rule implies ($d = m = 1$):

$$v'(x) = f_x'(x, y(x)) + f_y'(x, y(x))\, y'(x) = f_x'(x, y(x))$$

Alright? Ok, let's come back to our economic problem and our **Bellman equation**:

$$V(k) = \max_{c,\ell \in \Gamma(k)} U(c) + \beta V(k')$$

where $\Gamma(k) = \{c, \ell\, s.t.\, c + k' \leq F(k, \ell) + (1 - \delta)k\} \subset \mathbb{R}_+ \times \mathbb{R}_+$, and $k' = g(k, c, \ell) = F(k, \ell) + (1 - \delta)k - c$. Here, the value function depends on the state k, i.e. capital stock, (in most macro models, it will be the wealth in one form or another) while it is optimal w.r.t. the controls c_t, ℓ_t. The future.

(1.) First, we need to derive the *FOC conditions* of the problem at time t to obtain the policy functions ($c = \mathcal{C}(k)$ and $k' = \mathcal{K}(k)$ and determined using the ... KKT theorem). Rewrite the inequality constraint and Lagrangian: everything is state dependent, all the variables should have the s^t, but we only specify Y_t and $\lambda_t(s^t)$ to make clear where the shock comes from and affect the rest of the system:

$$F(c, k', \ell) = c + k' - F(k, \ell) - (1 - \delta)k \leq 0 \qquad \mathcal{L}(c, k', \ell, \lambda) = U(c, 1 - \ell) + \beta V(k') - \lambda F(c, k, \ell)$$

The FOC yields the following equations (note that we won't rewrite the primal and dual feasibility and complementarity conditions that are exactly the same as above) deriving w.r.t. c, ℓ and k'. Note that again, the Lagrange multiplier is state-dependent λ. The Lagrangian :

$[c_t]$ $$\frac{\partial U(c^*, 1 - \ell^*)}{\partial c} = \lambda \times 1$$

$[k_{t+1}]$ $$\left\{\frac{\partial U(\cdot)}{\partial k'} = 0\right\} \quad \beta \frac{dV(k')}{dk} = \lambda_t \times 1$$

$[\ell_t]$ $$\frac{\partial U(c^*, 1 - \ell^*)}{\partial \ell} = \lambda\, \frac{\partial F(k, \ell)}{\partial \ell}$$

(combining 1 & 3) $$U_\ell(c^*, \ell^*) = U_c(c^*, 1 - \ell^*)\, F_\ell(k, \ell) \qquad MRS_{1-\ell/c} = MPL_\ell$$

(combining 1 & 2) $$U_c(c^*, \ell^*) = \beta V'(k'^*) \qquad MU_{\text{conso}} = MV_{\text{income}}$$

(2.) Now, to derive the *Envelope condition*, we need to obtain the derivative of V (w.r.t. k) (why? In the last equation we still have the term $\partial_k V(k)$ unknown). Coming back to the Bellman equation, let us obtain $V'(k)$. Here, we can "simply" use the Envelope theorem: we need to derive the value function w.r.t. the state variable, while it is already maximized over the three controls. *But*, there is a trick, in the sense that, when the budget constraint is binding, one of the control can be expressed as a function of the other (state and control)

variables:

$$c_t = \mathcal{C}(k_t, k_{t+1}, \ell_t) = F(k_t, \ell_t) + (1-\delta)k_t - k_{t+1}$$

This implies that, after using the chain-rule, the envelope condition is expressed as:

$$\partial_k V(k) = \frac{\partial U(c, 1-\ell)}{\partial c} \frac{\partial \mathcal{C}(k, k', \ell)}{\partial k} + \beta \frac{dV(k')}{dk'} \frac{\partial k'}{\partial k}$$

And *here*, we can remember the Envelope theorem, telling us that this last term is zero, because V is already maximized over k (because of the FOC). Therefore:

$$V'(k) = \frac{\partial U(c, 1-\ell)}{\partial c} \frac{\partial \mathcal{C}(k, k', \ell)}{\partial k} = U_c(c, 1-\ell) \times \big(F_k(k, \ell) + (1-\delta)\big)$$

In this very simple case, the marginal value of wealth (i.e. saving) today equals the marginal utility of present consumption times the gain of the transfer between the present and future consumption. Surprisingly this intuition holds in many macro models.

(3.) We can combine everything together. This condition holds at all time, and we can iterate ($t \rightarrow t+1$), and obtain:

$$V'(k') = U_c(c', 1-\ell') \times \big(F_k(k', \ell') + (1-\delta)\big) = U_c(\cdot) R_t$$

where can call $F_k(k', \ell') + (1-\delta)$ the gross return of capital (which is simply the production minus the depreciation), and replacing in the last FOC equation above :

$$U_c(c, 1-\ell) = \beta V'(k') = \beta U_c(c', 1-\ell')\,(1 + r_{t+1})$$

We got the ***Euler equation*** (intertemporal FOC)! This implies the asset pricing equation:

$$U'(c, 1-\ell) = \beta U'(c', 1-\ell') R_t = \beta U'(c', 1-\ell')\big(F_k(k', \ell') + (1-\delta)\big)$$

With the dynamic programming methods, that seems a lot of work for one simple (and almost already known) equation ... but, the good news is that this method is general and can be applied to all kind of problems (especially when associated with the KKT theorem for inequality constraints). In the macro sequences (in 1st or 2nd year), you will use the Dynamic programming approach to study macro models with incomplete markets, credit constraints, nominal rigidities, search & matching frictions, heterogeneous agents, etc. There will be two main methods: use the Bellman algorithm numerically (solving with an approximation of the value function in order to obtain the policy function) or using make clever assumption to be able to derive the model analytically.

Value function as a fixed point – Contraction mapping theorem

In the previous section, we were kinda loose on the existence and uniqueness of the value function $V(x)$ and the maximizer. The following conditions will insure rigorously this formulation.

Let us start with some introductory definitions

Definition 6.4 (Introductory definitions).

- *A function T mapping a metric space (usually in infinite dimension) into itself is called an operator.*
- *Let (S,d) be a metric space and $T : (S,d) \to (S,d)$ an operator. Then T is said to be β-*<u>Lipschitz</u> *with modulus β if there is a number $\beta \in \mathbb{R}$ s.t.*

$$\forall y_1, y_2 \in S, \qquad d(T(y_1), T(y_2)) \leq \beta\ d(y_1, y_2)$$

- *An operator is said to be* <u>contraction</u> *with modulus β if it is β-Lipschitz $\beta \in (0,1)$*

Theorem 6.6 (Contraction mapping theorem).
Let (S,d) be a complete metric space and let T be a contraction mapping with modulus β. Then

1. *T has a unique fixed point y^* in S*
2. *For any $y_0 \in S$, the sequence $y_{n+1} = T(y_n)$ started at y_0 converges to y^* in metric d.*

Corollary 6.7.
Let S' be a closed subset of S. If $TS' \subset S'$, then $y^* \in S'$. Moreover, if $S'' \subset S'$ and $TS' \subset S''$, then $y^* \in S''$.

Theorem 6.8 (Blackwell condition).
Let T be an operator on a metric space (S, d^∞) where S is a space of functions with domain X and d^∞ is the supremum distance. Then T is a contraction mapping with modulus β if:

1. *(Monotonicity) $\forall f_1, f_2 \in S$,*

$$f_1(x) < f_2(x), \forall x \implies Tf_1(x) < Tf_2(x), \forall x$$

2. *(Discounting) For any function $f \in S$ and a positive real number $\alpha > 0$,*

$$T(f + \alpha) \leq Tf + \beta\alpha.$$

Hence T is a contraction in the space of functions (S, d^∞).

Let us recall the Bellman equation :

$$V(x) = \max_{c \in \Gamma(x)} F(x,c) + \beta V(g(x,c))$$

One can actually express the Bellman equation as a mapping from a future value $V\big(g(x, c(x))\big)$ to a present value $V(x)$

Proposition 6.9.

Define our Bellman operator T *as:*

$$TV(x) = \max_{c \in \Gamma(x)} F(x,c) + \beta V\big(g(x,c)\big)$$

This is an operator from the set of bounded continuous functions $\mathcal{C}_b(\mathbb{X})$ into itself.

Let us check that the Contraction mapping theorem (CMT) applies:

Let $\mathcal{C}_b(\mathbb{X})$ be the space of bounded, continuous function, $V(x) \in \mathcal{C}_b(\mathbb{X})$ for all $x \in X$. Use our definition of the Bellman operator and suppose $\widetilde{v}$ is bounded. For CMT to apply, we need that (i) $T\widetilde{v}$ is bounded and continuous; (ii) T is a contraction.

- TV is bounded:
  - Either, period return $F(x,y)$ is bounded—$\sup_{x,y \in \Gamma(x)} F(x,y) \leq \bar{F}$
  - Or, state space $\mathbb{X}$ is bounded—e.g. there exists some $\bar{x} > 0$ such that

    $$x' = \sup_{u \in \Gamma(x)} g(x,y) < x \qquad \forall x > \bar{x}$$

    and define $X = [0, \bar{x}]$ as the state space.
- TV is continuous
  - F varies continuously with x and y
  - $\Gamma(x)$ varies continuously with state (i.e. Γ is continuous as a correspondence)– this follows from the fact that the constraints are continuous (usually)
- T is a contraction: Check Blackwell's sufficient condition (T monotone and discounts).

We therefore established that CMT applies, which means that

- Principle of optimality holds; i.e. V solve the *eq.* ($P_{B.e.}$)
- T has a unique fixed point V such that $TV = V$.
- $c(x) \subseteq \Gamma(x)$ is compact-valued and upper hemicontinuous.

Let us check the other conditions on our primitives and see what it buys us to use a stronger version (on a smaller set) of the contraction mapping theorem.

Monotonicity:

- Requirement 1. $\Gamma(x)$ monotone (i.e. $x' \geq x \Rightarrow \Gamma(x) \subseteq \Gamma(x')$). This usually holds by the constraints as $\Gamma(x) = [0, \psi(x)]$ for some function ψ is strictly increasing.
- Requirement 2. $F(x,y)$ weakly increasing in x—let $\widehat{\mathcal{C}}(\mathbb{X}) \subset \mathcal{C}_b(\mathbb{X})$ be the set of weakly increasing functions (so the set is closed). Want to show that $\widetilde{v} \in \widehat{\mathcal{C}}(\mathbb{X}) \Rightarrow T\widetilde{v} \in \widehat{\mathcal{C}}(\mathbb{X})$—i.e. $T : \widehat{\mathcal{C}}(\mathbb{X})(X) \to \widehat{\mathcal{C}}(\mathbb{X})$. Usually holds for most profit or utility functions with local non-satiety.

⇒ If $F(x,y)$ is strictly increasing in x, then $T\widetilde{v}$ is strictly increasing (i.e. the fixed point v lies in the interior of $\widehat{\mathcal{C}}(\mathbb{X})$).

Concavity:

- Let $\widetilde{\mathcal{C}}(\mathbb{X}) \subset \mathcal{C}_b(x)$ be the set of functions that are weakly concave (so the set is closed). We want to show that $\widetilde{v} \in \widetilde{C} \Rightarrow T\widetilde{v} \in \widetilde{C}$—i.e. $T : \widetilde{\mathcal{C}}(\mathbb{X}) \rightarrow \widetilde{\mathcal{C}}(\mathbb{X})$.
- Requirement 1. $F(x,y)$ weakly concave in (x,y), again usually profit (with constant or decreasing returns) or utility functions (with some weakly positive risk aversion)
- Requirement 2. $\Gamma(x)$ is convex. Usually holds for most inequality constraints (with the right sign!)

⇒ If F is strictly concave in x, then $T\widetilde{v}$ is strictly concave (i.e. the fixed point v lies in the interior of $\widetilde{\mathcal{C}}(\mathbb{X})$). Then, there exists a unique maximiser which is continuous in x (i.e. the optimal policy correspondence G is, in fact, a single-valued function).

Differentiability of V at $\hat{x}$

- Requirement 1. v is strictly concave.
- Requirement 2. $\hat{x} \in \text{int}(X)$.
- Requirement 3. $c(\hat{x}) \in \text{int}(\Gamma(\hat{x}))$.
- Requirement 4. F is differentiable in x.
- We cannot use the method we used before since the set of differentiable function is not closed (with respect to the sup norm). The idea is to construct a strictly concave, differentiable W that lies everywhere below v and $w(\hat{x}) = v(\hat{x})$. Then W has the same supporting hyperplane as v so that v is differentiable at $\hat{x}$.[4] Fixing $\hat{y} = g(\hat{x})$ and if $\hat{y}$ is feasible in the neighbourhood of $\hat{x}$ (i.e. $\hat{y} \in \text{int}(\Gamma(\hat{x}))$), we can use $w(x) = F(x,\hat{y}) + \beta v(\hat{y})$. Since $\hat{y}$ maximises at $\hat{x}$,[5]
$$w(x) \leq w(\hat{x}) = v(\hat{x}), \ \forall x \neq \hat{x}.$$
Notice that $\hat{y}$ may not be feasible in the neighbourhood of $\hat{x}$ if v has a kink at $\hat{x}$. In such a case, the inequality above may not hold.
- Differentiability of v does not give differentiability of $c(\cdot)$. That would require twice differentiability of v. The proof is a "jungle" but it does require at least twice differentiability of F in x.

---

[4] At $x = \hat{x}$, the slope of w and v coincide:
$$\left.\frac{dw(x)}{dx}\right|_{x=\hat{x}} = F_x(\hat{x},\hat{y}) + \beta v'(\hat{y}) = v'(x).$$

[5] By construction, $w(\hat{x}) = v(\hat{x})$. Finally,
$$\begin{aligned} v(\hat{x}) &= \max_{y \in \Gamma(\hat{x})} F(\hat{x},y) + \beta v(y) = F(\hat{x},\hat{y}) + \beta v(\hat{y}) \\ &\geq F(x,\hat{y}) + \beta v(\hat{y}) = w(x) \end{aligned}$$
for all x in the neighbourhood of $\hat{x}$ such that $\hat{y}$ is feasible.

Hence we can use the contraction mapping to a much smaller set and prove result about the value function! This is useful for studying comparative statics.

Lastly it is good to verify Blackwell's conditions at least once:

Blackwell's conditions:

- (monotonicity) If $v_1(x) < v_2(x) \forall x$, then

$$\begin{aligned} Tv_1 &= \max_{c \in \Gamma(x)} F(x,c) + \beta v_1(g(x,c)) \\ &= F(x,c^*) + \beta v_1(g(x,c^*)) \\ &\leq F(x,c^*) + \beta v_2(g(x,c^*)) \\ &\leq \max_{c \in \Gamma(x)} F(x,c) + \beta v_2(g(x,c)) = Tv_2 \end{aligned}$$

- (discounting)

$$T(v+\alpha) = \max_{c \in \Gamma(x)} F(x,c) + \beta\big[v(g(x,c)) + \alpha\big] = Tv + \beta\alpha$$

A corollary of the contraction mapping theorem provides a tool to characterize the value function.

Corollary 6.10 (Characterization of a value function)**.**
In particular, let $\hat{F}(x,x') = F(x,c(x,x'))$ where $c(x,x')$ is implicitly defined by the FOC and by $x' = g(x,c(x,x'))$. Then:

- *If $\hat{F}(x,x')$ is (strictly) monotone in x then the fixed point $V(x)$ is also (strictly) monotone in x.*
- *If $\hat{F}(x,x')$ is (strictly) concave in x, then the fixed point V is also (strictly) concave in x.*

Dynamic programming with stochastic dynamics

We study the control problem with risk eq. (P_{c-s-r}). To this purpose we study the Bellman equation of the form:

$$v(x,z) = \max_{y \in \Gamma(x,z)} \left\{ F(x,y,z) + \beta \mathbb{E}\left[v\left(y,z'\right) |z \right] \right\}, \tag{5}$$

The definitions of the objects are the same as above, $x \in \mathbb{X}$ are the endogenous state variables while $z \in \mathbb{Z}$ are the exogenous (random) state variables. Note that the notations of this section[6] are slightly different of the ones above: the shock ε is now z and the control c is now denoted y. We will assume it follows a time-homogenous/Stationary Markov chains.

Recap on Markov chain Recall: a stochastic process is a stationary Markov chain if:

$$\mathbb{P}(z_{t+1} = s_j \mid z_0, \ldots, z_{t-1}, z_t = s_i) = \mathbb{P}(z_{t+1} = s_j | z_t = s_i) = q_{i,j}$$

$$\mathbb{P}(z_{t+1} = s_j | z_t = s_i) = \mathbb{P}(z_t = s_j | z_{t-1} = s_i), \ \forall t. \qquad \Rightarrow \qquad q_{t,i,j} = q_{t-1,i,j} \equiv q_{i,j}$$

Where the first condition is the Markov property, and second the time-homogeneity. Markov process fits perfectly the recursive nature of dynamic programming. Note that we can always redefine the state space and express n-order Markov process as first-order (at the price of enlarging the state-space, cf. example above). There are two basic cases to consider.

Discrete shocks The State space $\mathbb{Z} = \{z_1, z_2, \ldots\}$ is countable and transition probabilities are described by a transition matrix

$$Q = [q_{ij}], \ q_{ij} \geq 0, \ \sum_{j=1}^{\infty} q_{ij} = 1, \ \forall i.$$

The transition probability q_{ij} describes the probability of state z_j occurring in the next period, given current state of z_i.

Continuous shocks $\mathbb{Z} \subseteq \mathbb{R}^m$ is a closed rectangle In other words, $\mathbb{Z}$ is a product of closed intervals. For each $z \in \mathbb{Z}$, let $q(z'|z)$ be the density for the shocks next period (in the sense that the measure $P_{Z_t|Z_{t-1}=z}(dz')$ has a density w.r.t to the Lebesgue measure, c.f. our notation above).

In this case, we need an additional assumption to ensure that the value function is continuous.

Definition 6.5 (Feller property).
The conditional density $q(z'|z)$ is said to have the Feller property if, for any fixed $z' \in \mathbb{Z}$, $q(z'|z)$ is continuous in z. Let $f(z)$ be a continuous function defined on $\mathbb{Z}$ and that $q(z'|z)$

[6] That follow more closely the notations of N. Stokey in her class of Theory of Income

satisfies the Feller property. Then,

$$(\mathbb{T}f)(z) = \int_{\mathbb{Z}} f(z') \, q(z'|z) \, dz' = \mathbb{E}[f(z')|z]$$

is continuous on $\mathbb{Z}$. Note that $\mathbb{E}[f(z')|z]$ is a conditional expectation w.r.t Z taken at value z. Thus, the Feller property, in addition to the conditions required in the discrete dynamic programming case (F varies continuously with x and y, and $\Gamma(x,z)$ varies continuously with x for any fixed z), guarantees that the Bellman operator is continuous. Note that, if $\mathbb{Z}$ is discrete, then the Feller property is trivially satisfied. For this continuous Bellman operator to be well-defined, we also need it to be bounded and for it to be a contraction (as in the deterministic case).

The arguments for monotonicity, continuity, and differentiability of v and G in x are the same as in the discrete case (the must hold for any fixed $z \in \mathbb{Z}$).

To show that $v(x,z)$ is increasing in z (not x!), we assume, in addition:

- F is increasing in z;
- $\Gamma(x,z)$ is monotone in z (i.e. $\hat{z} \geq z \Rightarrow \Gamma(x,z) \subseteq \Gamma(x,\hat{z})$)
- Q transition function is monotone; i.e. conditional expectation of $f(\cdot)$, $\mathbb{T}f$, is increasing in z if $f(z)$ is increasing in z. In the case $z \in \mathbb{R}$, this condition is equivalent to first-order stochastic dominance; i.e. $\hat{z} \geq z \Rightarrow F(z'|\hat{z}) \leq F(z'|z)$.

We can interpret the last condition as a requirement that ensures that a higher shock today implies a "better" distribution tomorrow.

We often set up the problem of the following form:

$$v(x,z) = \max_{y \in \Gamma(x,z)} \left\{ F(x,y,z) + \beta \int_{\mathbb{Z}} v\left[g(x,y,z'), z'\right] Q(z, dz') \right\},$$

where ϕ is the law of motion of the endogenous state variable: $x_{t+1} = g(x_t, y_t, z_{t+1})$. For example, we can think of x as the current level of assets, y as the amount of saving and z' as the next-period interest rate which, together with the current level of assets and savings, determine the next-period level of assets.

Controlled dynamics, Kolmogorov forward and distribution over states

This section is forthcoming

But as an extreme summary remember that if the state z follows a Markov process over the state space $\mathbb{Z}$ and x is a controlled process such that $x' = g(x, c^\star, z') = g(x, c(x,z), z')$ we can write the vector $w = (x,z)$ as a Markov chain in $\mathbb{X} \times \mathbb{Z}$, with a transition matrix

$$\mathcal{Q} = \{q_{i,j}\}_{i,j} \quad \text{such that} \quad \forall i,j, \; q_{i,j} = \mathbb{P}((X_{t+1}, Z_{t+1}) = w_j | (X_t, Z_t) = w_i)$$

Moreover, one can compute the distribution over states

$$g_t(x,z) = P_{X_t,Z_t}(x,z) = \mathbb{P}\big((X_t,Z_t)=(x,z)\big)$$

and the law of motion of the distribution (also called the Kolmogorov Forward in continuous time) as

$$g_{t+1}(x,z) = \int_{x,z} g_t(\widetilde{x},\widetilde{z})\mathcal{Q}((d\widetilde{x},d\widetilde{z}),x,z)$$

Full Example : RBC model

Should government respond proactively to business cycles? The canonical RBC model says no—i.e. business cycles are an outcome of efficient response to (stochastic) productivity shocks.

Example 6.3 (RBC model)**.**

The setup of the control model is as follow:

- *Representative household with preferences given by* $u(c,h)$*.*
- *Representative firm with CRS technology* $y = zf(k,h)$*.*
- *Stochastic process* $z \in \mathbb{Z}$ *with transition density* $q(z'|z)$*. Assume that the transition function has the Feller property and that it is monotone.*
- *Since the shocks are aggregate shocks, households cannot insure against shocks even if the markets were complete.*
- *General observation: Shocks tend to be very persistent—AR(1) coefficient of around* 0.9 *to* 0.95*.*
- *No taxes and no government —we can solve the planner's problem.*

The Bellman equation is given by

$$\begin{aligned} v(k,z) = \max_{c,h,k'} \quad & u(c,1-h) + \beta \int_{\mathbb{Z}} v(k',z')\, q(z'|z)\, dz' \\ s.t. \quad & c + k' - zF(k,h) - (1-\delta)k \le 0. \end{aligned}$$

The goal of RBC model was to create a model for which (c,h,k') move together with z, which can be interpreted as total factor productivity shock. Note that this is a *supply-side* shock. Why not a demand shock? The problem is that demand shocks will not produce a result in which c and k' move together—since income is not higher with higher demand, higher consumption implies lower saving, i.e. k'.

FOC and EC

Denote λ Lagrange multiplier on the feasibility constraint, the first-order conditions are

$$\begin{aligned} \{c\} \qquad & u_c(c,1-h) = \lambda \\ \{h\} \qquad & u_\ell(c,1-h) = \lambda z F_h(k,h) \\ \{k'\} \qquad & \beta\mathbb{E}\left[v_k(k',z')\,|z\right] = \lambda \end{aligned}$$

The envelope condition is

$$v_k(k,z) = \lambda\left(z\,F_k(k,h) + (1-\delta)\right)$$

Comparative statics with Inelastic labour supply

We focus on the interior solution case. To start with, suppose that labour supply is inelastic; i.e. $h = \bar{h}$. Then, combining the 1st and 3rd FOCs gives the optimality equation

$$\begin{aligned}\beta\mathbb{E}\left[v_k\left(k',z'\right)|z\right] &= u_c(c,1-h) \\ &= u_c\left(zF(k,h) + (1-\delta)k - k', 1-h\right).\end{aligned}$$

As usual this type of Euler equation equates the marginal (discounted) expected value from investment with marginal utility from consumption.

From this equation, we can see that k' is given by the intersection of two functions, expressed as functions of y;

$$\beta\mathbb{E}\left[v_k\left(y,z'\right)|z\right] = u_c\left(e^z F(k,h) + (1-\delta)k - y\right).$$

Given the assumptions, we know that v is a strictly increasing, concave function. Thus, v_k is positive and strictly decreasing in y. On the other hand, u_c is strictly positive and increasing in y. Thus, there can be at most one solution. The solution exists since we are maximising a continuous function in a compact set. Full Inada conditions would guarantee that the solution is interior.

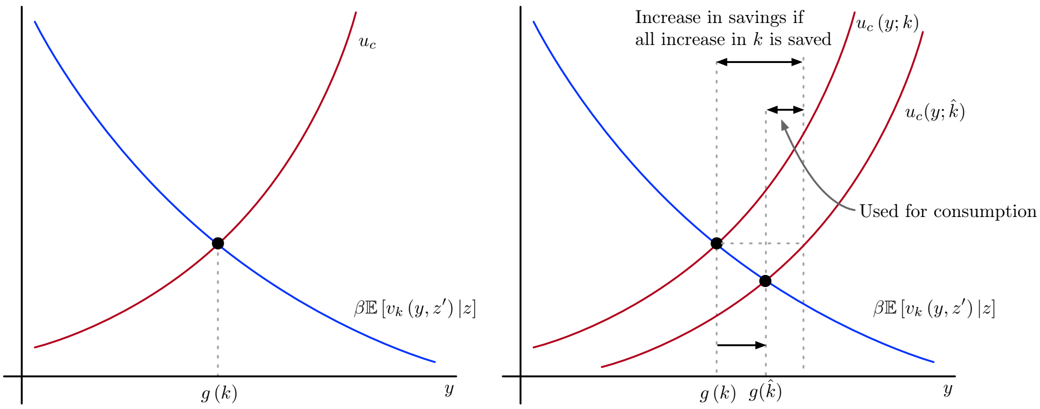


Figure 1: **Policy function $y \equiv k'$ as a function of k**

What happens to $k' = g(k)$ (i.e. future capital) if current capital k, increases from k to $\hat{k}$?

Then only the u_c curve moves—since F is increasing in k and utility function is strictly concave, a higher k shifts the u_c curve to down/right. As the figure shows, $g(k)$ increases to $g(\hat{k})$. However, notice that the change in $g(k)$ is less than the horizontal movement of the u_c curve. If household were to save all of the increase in k (in the form of higher $g(k)$), then the movement in the curve would coincide with the movement of $g(k)$. The fact that it moves by less means that households are, in fact, consuming some of the higher stock of capital today. Letting $\Delta k = \hat{k} - k$, we have that

$$(zF_k + 1 - \delta)\,\Delta k < \Delta g\,(k)\,.$$

What happens to $k' = g\,(k)$ if current shock $(\hat{z})$ were higher?

Since u_c is increasing in z, (same as an increase in k), the u_c curve moves right/down. However, with a higher z, $\beta\mathbb{E}\,[v_k|z]$ could also shift (assuming shocks are not iid).

Empirical data suggests that positive technology shocks such as those represented by y has a positive effect (i.e. a higher shock implies a better distribution tomorrow), implying that CDFs (i.e. transition functions) are ordered by FOSD. This, in turn, implies that v is increasing in z; however, we do not know how v_k changes with z (nor do we know how the policy function changes with z). This is because a "better" distribution has two effects.

On the one hand, a better distribution in the future incentivizes households to save *less*, but, on the other hand, a higher shock today means more income today, which incentivizes households to save *more*. The net effect is therefore ambiguous.

Data suggests that a positive shock increases both savings and capital, meaning that the movement of the $\beta\mathbb{E}\,[\cdot]$ curve should be small (either to the left, or to the right).

Full Example : Aiyagari model

The consumption-saving problem of Huggett and Bewley, and the General equilibrium version of Aiyagari is the main benchmark for heterogeneous agents models, that became ubiquitous with the development of computational methods.

Example 6.4 (Aiyagari model)**.**
The setup of the control model is as follow:

- *Representative household with preferences given by $u\,(c,h)$.*
- *Two states: wealth a and labor productivity z ; control consumption: c*
- *Idiosyncratic fluctuation in z (Markov process)*
- *Incomplete market: Households cannot insure against shocks*
- *State constraint (no borrowing) $a_t \geq \underline{a}$*
- *Representative firm with CRS technology $y = f\,(k,h)$.*
- *Interest rate : $r_t = f_k(K,L) - \delta$ & wage $w_t = f_h(K,L)$*
- *Influence of distribution of agents through the aggegate capital (which is the average of savings)*

$$S_t := \sum_z \int_{\underline{a}}^{\infty} a\, g_t(da, z_j) = K_t$$

The control problem (in sequential form) is:

$$\max_{c_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad\qquad c_t + a_{t+1} = z_t w_t + r_t\,(1 + a_t)$$

This model – as all heterogeneous agents problems includes:

- ▷ A Bellman equation: backward in time
  How the agent value/decisions change when distribution is given
- ▷ A Law of Motion of the distribution: forward in time
  How the distribution changes, when agents control is given
- ▷ These two relations are coupled
  Through firm pricing (r_t & w_t) $\Rightarrow$ need to look for an equilibrium fixed point
  s.t. Supply of asset = demand of capital

These three equations are the following:

$$v_t(a,z) = \max_{c,a'} u(c) + \beta \mathbb{E}\big[v_{t+1}(a',z') \big| \sigma(z)\big]$$
$$s.t. \qquad c + a' = z w_t + r_t\,(1+a) \qquad a' \geq \underline{a} \quad \Rightarrow \quad a'^{\star} = \mathscr{A}(a,z)$$

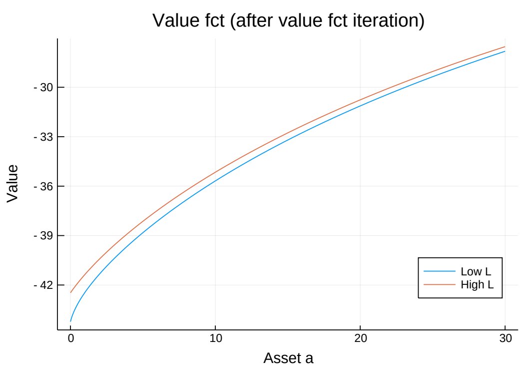
$$\forall\ \widetilde{A} \subset [\underline{a}, \infty) \qquad g_{t+1}(\widetilde{A}, z') = \sum_z \pi_{z'|z} \int \mathbb{1}\{\mathscr{A}(a,z) \in \widetilde{A}\}\, g_t(da, z)$$

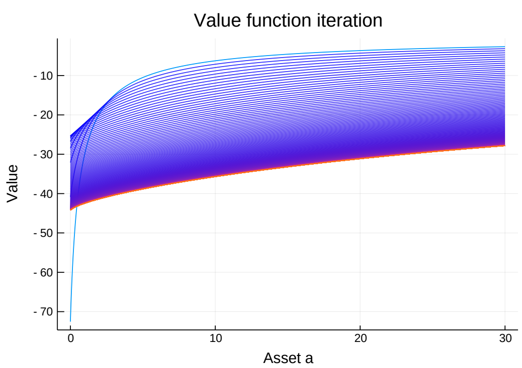
$$S_t(r) := \sum_z \int_{\underline{a}}^{\infty} a\, g_t(da, z_j) = K_t(r)$$

Usually, this needs to be solved on the computer. In the following we take the simple case where $z \in \{\underline{z}, \overline{z}$ two values (that can correspond to employed/unemployed). We also take the most standard functional forms for utility and production.

Value function

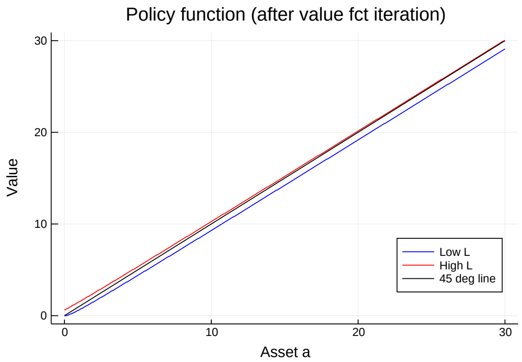
In the next graph, we see how the Value function converges using value function iteration (i.e. repeatedly applying the Bellman operator), starting from an improved guess $v_0(a,z) = \frac{1}{1-\beta} u((1+r)a + zw)$

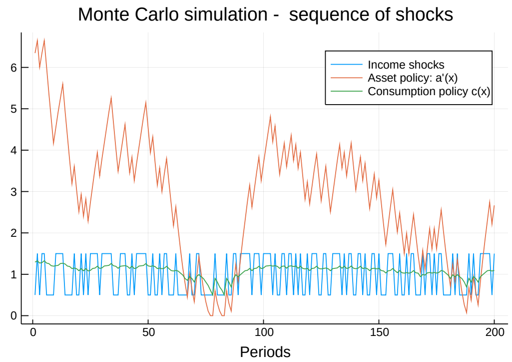




<u>*Optimal policy and a closer look to a model simulation*</u>

The policy function for the future asset is displayed in the following graph. The makes points to realize here are that : (i) future asset $a = g(a, z)$ is higher for higher a and higher z (red is higher than blue), (ii) the agent with lower $\underline{z}$ *hits* the borrowing constraints for low a and (iii) the agent with higher $\overline{z}$ have a slope of policy lower that 45 degree line (that insure the fact that the controlled dynamics for a_t stays in a compact ergodic set).





On the RHS we also show that a sequence of shocks for z (taken randomly) and how it affects $c = c(a, z)$ and $a' = g(a, z)$. We start initially from an asset $a_0 = 7$ but the evolution is similar for any asset above $a = 2$: in particular, for high assets, the first 50 periods always feature a drop in the saving (measured by the future stock of capital) because of consumption smoothing.

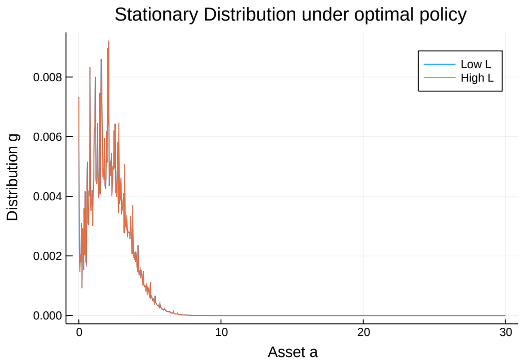
As we see the consumption is very smooth compared to income and this is due to the ability of household to save and self-insure by accumulating/decumulating assets. Note that the consumption may feature significant drops when assets are very low: credit constrained household are usually more hand-to-mouth (high MPC) than wealthier ones.

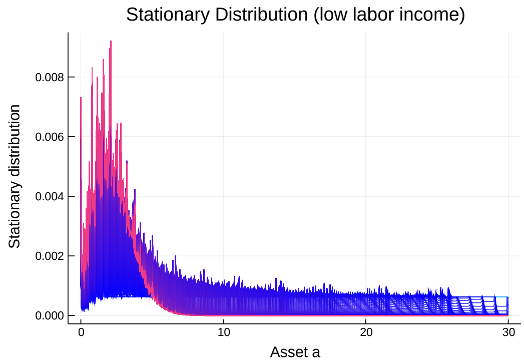
Out of a given sequence of shocks (says out of 200 periods), we can compute the fraction of period when households are credit constrained by Monte Carlo simulation (i.e. simulating n_MC = 1000 or 10000 times the model and averaging the results). This share will be very different for different initial wealths, and we show the result for three values. For $a = 7$, the fraction is 1.131%, for $a = 2$, the share is 1.452%. For $a = 0$ (starting at the constraint), the share rises to 2.228%. This share is fairly small and may be higher in the Aiyagari's calibration (mainly due to autocorrelation of shocks and lower "low level" income ℓ_l).

The asset holding usually do not go too high – around 3 or 4 – and may be caused by positive sequence of income shocks. The precautionary saving are relatively small (even more than in Aiyagari's paper where he shows that in such setting precautionary saving are smaller than expected). This may -again- due to the simple structure of the risk and the i.i.d. assumption. The capital stock is still higher than in the neoclassical model.

Stationary distribution

To show the convergence of the stationary distribution, the next plot display the convergence from an initial uniform distribution (in light blue) to red-color stationary distribution which is the fixed point of the mapping for the evolution of the distribution. More precisely, this is the solution of the Kolmogorov forward equation for this controlled process.

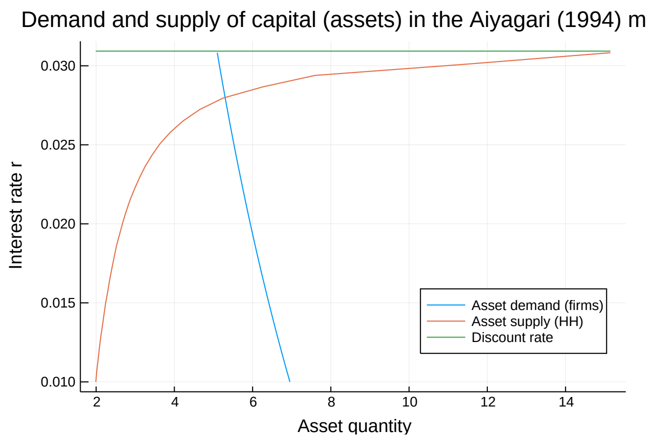




As we can see, this convergence is rather chaotic due to the discretization/approximation, the discrete time and the non-sparse optimal policy. The distribution nonetheless converges by a standard result of Markov chains (this chain is ergodic and the ergodic set represent a set which is bounded above by a_{max}). Hence, the stationary distribution over this set converges even starting for any initial distribution. This remark is observed in the previous plot – where the discretization error (and transportation of measure from high value to low value) is soon overcome by the exponential convergence result.

Market clearing

As we mention above the distribution of agent has an impact on the aggregate prices and production. In the next figure, we plot a similar figure as in Aiyagari (1994), which displays the supply of capital saved by household (intertemporal smoothing and precautionary saving). This supply of capital $A(r)$ is represented in red while the demand by firms (standard neoclassic model capital demand) is drawn in blue (and is really steep). Note that the price associated to the supply of capital (price) is bounded above by the discount factor $\rho = 1/\beta - 1$



7 Control theory in continuous time

7.1 Sequential approach and Pontryagin Maximum principle

We use the control-state formulation. For this, we have the instantaneous return function F that depends on the state vector $x \in X \subseteq \mathbb{R}^n$ and a control vector $c \in \Gamma(x) \subseteq \mathbb{R}^k$ (k need not equal n). The problem is

$$\begin{aligned} V^*\left(x_0\right) \coloneqq \max_{\{c(t)\}_{t=0}^{\infty}} \quad & \int_0^\infty e^{-\rho t} F\left(x\left(t\right), c\left(t\right)\right) dt \\ s.t. \quad & \dot{x}\left(t\right) = g\left(x\left(t\right), c\left(t\right)\right), \ \forall t \geq 0 \\ & c\left(t\right) \in \Gamma(x(t)), \ \forall t \geq 0, \\ & x_0 \text{ given.} \end{aligned}$$

We will study a procedure to obtain necessary and sufficient conditions for an optimum. This requires some regularity conditions.

Definition 7.1 (Hamiltonian)**.**
Let λ be a vector on $\mathbb{R}^n$ of co-state *variables. H the* current-value Hamiltonian *function is defined as*

$$H\left(x, c, \lambda\right) \coloneqq F\left(x, c\right) + \lambda g\left(x, c\right).$$

Theorem 7.1 (Pontryagin Maximum Principle)**.**
The following conditions are necessary (under regularity assumptions) and sufficient (under regularity and convexity assumptions) for the path of x and c to be optimal:

$$\begin{aligned} & H_c\left(x\left(t\right), c\left(t\right), \lambda\left(t\right)\right) = 0 & [FOC] \\ & \qquad \Rightarrow \qquad F_c\left(x, c^\star\right) + \lambda g_c\left(x, c^\star\right) = 0 \\ & H_x\left(x\left(t\right), c\left(t\right), \lambda\left(t\right)\right) = -\dot{\lambda}\left(t\right) + \rho\lambda\left(t\right) \\ & \qquad \Rightarrow \qquad \dot{\lambda} = \lambda\big(\rho - g_x\left(x, c^\star\right)\big) - F_x\left(x, c^\star\right) \\ & H_\lambda\left(x\left(t\right), c\left(t\right), \lambda\left(t\right)\right) = \dot{x}\left(t\right) \\ & \qquad \Rightarrow \qquad g\left(x\left(t\right), c^\star\left(t\right)\right) = \dot{x}\left(t\right), \end{aligned}$$

for all $t \geq 0$.

The state variable(s), x, has an initial value of x_0 and the co-state variable(s), $\lambda\left(t\right)$, have a boundary condition—the Transversality Condition—given by

$$\lim_{T \to \infty} e^{-\rho T} \lambda\left(T\right) x\left(T\right) = 0.$$

The initial value of the co-state variable, $\lambda(0)$, is not predetermined and it has to be solved as part of the system.

The interpretation of co-state is that $e^{-\rho t}\lambda(t)$ is the marginal value at time zero of an infinitesimal increase in the state x at time t.

The first condition says that the derivative of the Hamiltonian with respect to the control is zero—this gives the optimal choice of c (one for each control variable). The second condition shows a law of motion for the co-state, which represents how the shadow value/marginal value of the control moves over time. Finally, the last condition is the feasibility condition.

Heuristic proof

The Lagrangian

Let us form the Lagrangian, using $e^{-\rho t}\lambda(t)$ for the multiplier of $\dot{x}(t) = g(x(t), c(t))$:

$$\mathcal{L}(x, c, \lambda) = \lim_{T\to\infty}\left(\int_0^T e^{-\rho t}F(x(t), c(t))\,dt + \int_0^T e^{-\rho t}\lambda(t)\left[g(x(t), c(t)) - \dot{x}(t)\right]dt\right).$$

We want to maximize $\mathcal{L}$ with respect to x and c, and minimize with respect to λ.[7] First, consider the following term, and use integration by parts to obtain:

$$\int_0^T e^{-\rho t}\lambda(t)\,\dot{x}(t)\,dt = \left[e^{-\rho t}\lambda(t)\,x(t)\right]_0^T - \int_0^T\left[-\rho e^{-\rho t}\lambda(t) + e^{-\rho t}\dot{\lambda}(t)\right]x(t)\,dt.$$

We therefore have

$$\begin{aligned}\mathcal{L}(x, c, \lambda) = &\lim_{T\to\infty}\int_0^T e^{-\rho t}F(x(t), c(t))\,dt \\ &+ \lim_{T\to\infty}\int_0^T e^{-\rho t}\left[\lambda(t)\,g(x(t), c(t)) - \rho\lambda(t)\,x(t) + \dot{\lambda}(t)\,x(t)\right]dt \\ &- \lim_{T\to\infty}\left[e^{-\rho t}\lambda(t)\,x(t)\right]_0^T.\end{aligned}$$

Maximizing with respect to x

Since $\mathcal{L}(x, c, \lambda)$ has to be maximised by x, the first-order conditions with respect to $x(t)$ gives

$$\frac{\partial\mathcal{L}(x, c, \lambda)}{\partial x(t)} = e^{-\rho t}\left[F_x(x(t), c(t)) + \lambda(t)\,g_x(x(t), c(t)) - \rho\lambda(t) + \dot{\lambda}(t)\right] = 0.$$

Since $e^{-\rho t} > 0$, it follows that

$$\dot{\lambda}(t) = \rho\lambda(t) - F_x(x(t), c(t)) - \lambda(t)\,g_x(x(t), c(t)).$$

[7] Recall that the Lagrangian method is a max-min problem.

Note that

$$\frac{\partial H(x,c,\lambda)}{\partial x(t)} = H_x(x(t),c(t),\lambda(t)) = F_x(x(t),c(t)) + \lambda(t)\, g_x(x(t),c(t))$$

so that we can rewrite $\dot{\lambda}(t)$ as

$$\dot{\lambda}(t) = \rho\lambda(t) - H_x(x(t),c(t),\lambda(t)). \tag{6}$$

Maximizing with respect to c

The first-order condition with respect to $c(t)$ is

$$\frac{d\mathcal{L}(x,c,\lambda)}{dc(t)} = e^{-\rho t}\left[F_u(x(t),c(t)) + \lambda(t)\, g_u(x(t),c(t))\right] = 0.$$

Since

$$\frac{\partial H(x,c,\lambda)}{\partial c(t)} = H_c(x(t),c(t),\lambda(t)) = F_c(x(t),c(t)) + \lambda(t)\, g_c(x(t),c(t)),$$

we therefore have that

$$H_c(x(t),c(t),\lambda(t)) = 0. \tag{7}$$

Full Example : Ramsey-Cass-Koopmans Neoclassical growth model (NCG)

We now analyze the neoclassical model using the Pontryagin Maximum Principle. The period-return function is $U(c)$ and the law of motion is given by $\dot{k} = f(k) - \delta k - c$ (note that $k(0)$ is given). That is,

$$\begin{aligned} F(k,c) &= U(c), \\ g(k,c) &= f(k) - \delta k - c, \\ \Rightarrow H(k,c,\lambda) &= U(c) + \lambda\left(f(k) - \delta k - c\right). \end{aligned}$$

Then, the PMP applies and we have the necessary and sufficient (note that we have convexity of this problem because $U(\cdot)$ and $f(\cdot)$

$$\begin{aligned} H_c(\cdot) = 0 &\Rightarrow U'(c) = \lambda, \\ H_x = -\dot{\lambda} + \rho\lambda &\Rightarrow \dot{\lambda} = \lambda\left(\rho - \left(f'(k) - \delta\right)\right), \\ H_\lambda = \dot{x} &\Rightarrow \dot{k} = f(k) - \delta k - c \\ \lim_{T\to\infty} & e^{-\rho T}\lambda(T)\, k(T) = 0. \end{aligned}$$

We therefore have the following dynamic equations, along with a transversality condition

$$\dot{\lambda} = \lambda \left(\rho - \left(f'(k) - \delta\right)\right),$$
$$\dot{k} = f(k) - \delta k - c.$$

<u>*Phase diagram in (k, λ) space*</u>

To draw the phase diagram in (k, λ) space, note that

- $\dot{\lambda} = 0$: $f'(\bar{k}) - \delta = \rho$. In (k, λ) space, this is a vertical line. <u>*Dynamics:*</u> From $\dot{\lambda} = 0$ if $k > \bar{k}$, then $f'(k)$ is lower and $-f'(k)$ is higher so that $\dot{\lambda} > 0$. And if $k < \bar{k}$ then $\dot{\lambda} < 0$.
- $\dot{k} = 0$: $c = f(k) - \delta k$. Since $\lambda = U'(c)$, we have that $\lambda = U'(f(k) - \delta k)$. Note that $U' > 0$ and $U'' < 0$ so that $U'(\cdot)$ is a strictly decreasing transformation. Thus, when $f(k) - \delta k$ achieves its maximum (at $\hat{k}$ such that $f'\left(\hat{k}\right) = \delta$), $U'(f(k) - \delta k)$ is at its minimum. As $k \to 0$, $f(k) \to 0$ and $U'(f(k) - \delta k) \to \infty$. As $k > \hat{k}$, $f\left(\hat{k}\right)$ falls so that λ increases. These observations imply that $\dot{k} = 0$ locus is U-shaped in (k, λ) space. <u>*Dynamics:*</u> If $c > \bar{c}$, then $\dot{k} < 0$ and if $c < \bar{c}$, then $\dot{k} > 0$. Note that $c > \bar{c}$ represents points below the $\dot{k} = 0$ locus.

This gives the following phase diagram.

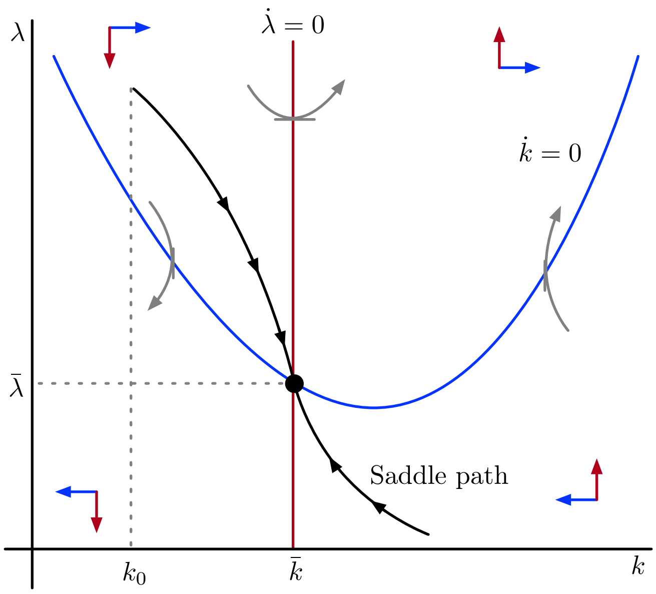


Figure 2: **Phase diagram in the (k, λ) space**

We see that low value of k corresponds to a higher value of λ. Does it make sense? Recall that λ is the marginal value of a unit of k and, at the optimal, marginal benefit from c is equated to λ, the marginal value of capital. In a similar way that higher c implies lower marginal benefit, a higher k implies lower marginal value of capital; i.e. it represents diminishing returns. Note also that a higher k implies higher production, and since c is a normal good (this is due to the fact that we have separable utility function), higher output implies higher income and c increases in every period.

Phase diagram in (k,c) space Since there is a one-to-one mapping between c and λ, we can also draw the phase diagram in (k,c) space. To do so, we can differentiate $H_u = 0$ condition with respect to time to obtain $\dot{\lambda}$ in terms of c:

$$U''(c)\,\dot{c} = \dot{\lambda}.$$

Then we can rewrite the second first-order condition as

$$\begin{aligned} U''(c)\,\dot{c} &= U'(c)\left(\rho - \left(f'(k) - \delta\right)\right) \\ \Rightarrow \frac{\dot{c}}{c} &= \frac{1}{-\frac{U''(c)c}{U'(c)}}\left(f'(k) - \delta - \rho\right), \end{aligned}$$

where $-\frac{U''(c)c}{U'(c)} = \sigma$ is the risk aversion/curvature of the utility, and $\frac{1}{\sigma} = -\frac{U'(c)}{U''(c)c}$ is the elasticity of intertemporal substitution, and $\dot{c}/c$ is the percentage change in c. Thus, we see that given $f'(k)$ and ρ, c is given by the preference. Notice also that, if $f'(k) - \delta = \rho$, then c is constant, and if $f'(k) - \delta > \rho$, then as U becomes linear – i.e. $U'' \to 0$ and $\frac{1}{\sigma} \to \infty$ – $\dot{c}$ grows larger so that you converge more quickly to the steady state (indeed the growth rate of consumption would rise up to ∞ with the IES growing: Household would perfectly substitute between present and future consumption, inducing large change in consumption over time depending on the relation between the rate of return on capital and the impatience/discount factor.

In fact, speed of convergence here is affected by both the curvature of the utility function as well as the curvature of production function as you will/might study with N. Stokey or F. Alvarez.

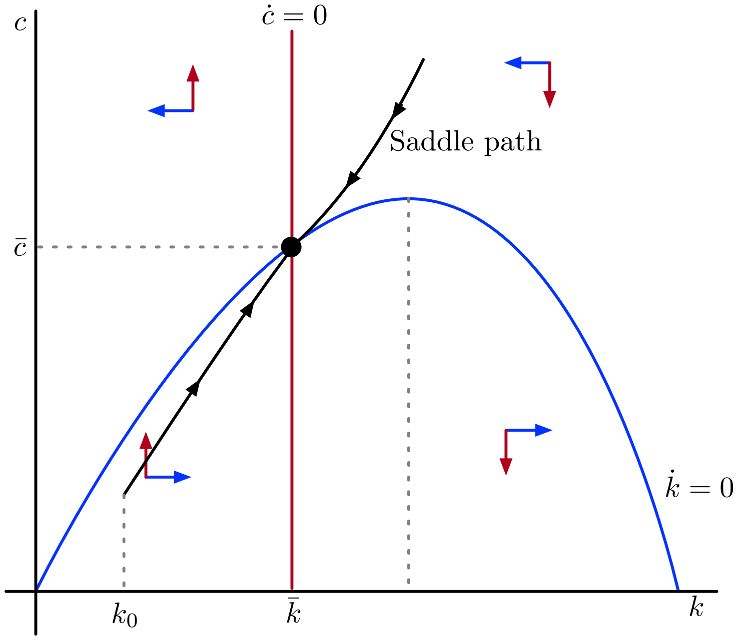


Figure 3: **Phase diagram in the (k,c) space**

Additional example: Hotelling's model of exhaustible resources

Example 7.1.

Consider a planner choosing an optimal path to extract some exhaustible resource. Let $\pi(q_t)$ be the flow payoff from flow extraction of a quantity q_t and the stock of resource S_t. The planner discounts the flow payoff at the interest rate r_t.

Given initial stock of the resource S_0, the problem can be formulated as:

$$\begin{aligned} V(S_0) = \max_{\{q\}_{t\geq 0}} & \int_0^{\infty} e^{-\int_0^t r_s ds} \pi(q_t) dt \\ s.t.\ & \dot{S}_t = -q_t \\ & S_t \geq 0 \\ & q_t \geq 0 \\ & S_0 \ given \end{aligned}$$

Let λ_t be the co-state, then the present-value Hamiltonian of the problem is given by:

$$H(S_t, q_t, \lambda_t) = e^{-\int_0^t r_s ds} \pi(q_t) + \lambda_t(-q_t)$$

The necessary conditions are given by:

$$\begin{aligned} H_q = 0 & \iff e^{-\int_0^t r_s ds} \pi'(q_t) - \lambda_t = 0 \\ H_S = -\dot{\lambda}_t & \iff \dot{\lambda}_t = 0 \\ H_\lambda = \dot{S}_t & \iff \dot{S}_t = -q_t \end{aligned}$$

Let the price be the marginal revenue – since we are supposed to be in a competitive market – then $p_t = \pi'(q_t)$, then we have the *Hotelling's rule* for exhaustible resource:

$$\dot{\lambda}_t = 0 \implies p_t = p_0 e^{\int_0^t r_s ds} \implies \frac{\dot{p}_t}{p_t} = r_t, \forall t$$

7.2 Recursive approach and Hamilton-Jacobi-Bellman equations

Recall the sequential formulation of the problem:

$$
\begin{aligned}
V^*(x_0) \coloneqq \max_{\{c(t)\}_{t=0}^{\infty}} \quad & \int_0^{\infty} e^{-\rho t} F(x(t), c(t))\, dt \\
s.t. \quad & \dot{x}(t) = g(x(t), c(t)), \ \forall t \geq 0 \\
& c(t) \in \Gamma(x(t)), \ \forall t \geq 0, \\
& x_0 \text{ given.}
\end{aligned}
$$

Exactly as this control problem is the limit of eq. (P_{c-s}), when the time step Δ goes to zero, in the following we will analyze the continuous-time Bellman equation as a limit of the discrete-time Bellman equation, as see above:

$$
V(x_t) = \max_{c_t \in \Gamma(x_t)} \left[\Delta F(x_t, u_t) + \frac{1}{1 + \Delta\rho} V(x_{t+\Delta}) \right]
$$

subject to

$$
x_{t+\Delta} = x_t + \Delta g(x_t, c_t).
$$

The continuous-time Bellman equation at the limit is the following:

Proposition 7.2 (Hamilton Jacobi Bellman equation)**.**
The continuous-time limit of the Bellman equation is called the Hamilton-Jacobi-Bellman equation and is defined as follow.

$$
\rho V(x) = F(x, c^*(x)) + V'(x)\, g(x, c^*(x))
$$

The very useful and intuitive proof for its construction is the following:

Proof

Notice that, if we simply take the limit as Δ goes to zero, we are simply left with $V(x_t) = V(x_t)$, which is not very useful. Using Taylor expansion (around x_t), we can write

$$
\begin{aligned}
V(x_{t+\Delta}) &= V(x_t + \Delta g(x_t, c_t)) \\
&= V(x_t) + V'(x_t)\, \Delta g(x_t, c_t) + o(\Delta g(x_t, c_t)),
\end{aligned}
$$

where $d(z) = o(z)$ means that $\lim_{z \to 0} d(z)/z = 0$. Then the Bellman equation is

$$
V(x_t) = \max_{c_t \in \Gamma(x_t)} \left[\Delta F(x_t, c_t) + \frac{1}{1 + \Delta\rho} \left(V(x_t) + V'(x_t)\, \Delta g(x_t, c_t) + o(\Delta g(x_t, c_t)) \right) \right]
$$

Multiplying both sides by the positive constant $1 + \Delta\rho$ yields

$$
(1 + \Delta\rho) V(x_t) = \max_{c_t \in \Gamma(x_t)} \left[(1 + \Delta\rho)\, \Delta F(x_t, c_t) + V(x_t) + V'(x_t)\, \Delta g(x_t, c_t) + o(\Delta g(x_t, c_t)) \right].
$$

We can move $V(x_t)$ inside the max to the left-hand side since it does not depend on c_t,

$$
\Delta\rho V(x_t) = \max_{c_t \in \Gamma(x_t)} \left[(1 + \Delta\rho)\, \Delta F(x_t, c_t) + V'(x_t)\, \Delta g(x_t, c_t) + o(\Delta g(x_t, c_t)) \right].
$$

Dividing both sides by Δ,

$$\rho V(x_t) = \max_{c_t \in \Gamma(x_t)} \left[(1 + \Delta\rho) F(x_t, c_t) + V'(x_t) g(x_t, c_t) + \frac{o(\Delta g(x_t, c_t))}{\Delta} \right].$$

Taking the limit as Δ goes to zero,

$$\rho V(x_t) = \max_{c_t \in \Gamma(x_t)} \left[F(x_t, c_t) + V'(x_t) g(x_t, c_t) \right],$$

where we implicitly assume that the limit as $\Delta \to 0$ of the max with respect to c_t is the same as the max with respect to c_t of the limit as $\Delta \to 0$. Removing the time indices, we obtain the continuous-time version of the Bellman equation,

$$\rho V(x) = \max_{c \in \Gamma(x_t)} \left[F(x, c) + V'(x) g(x, c) \right].$$

Under the regularity conditions, the max of the RHS can be characterized using the following first-order condition for c:

$$0 = F_c(x, c^*(x)) + V'(x) g_c(x, c^*(x)),$$

which defines the optimal decision rule $c^*(x)$. Thus, the following two equations summarize the dynamic programming problem:

$$\rho V(x) = F(x, c^*(x)) + V'(x) g(x, c^*(x))$$
$$0 = F_c(x, c^*(x)) + V'(x) g_c(x, c^*(x))$$

for all $x \in X$. Notice that these are two functional equations (i.e. solutions are functions). The functions V and c^* are both functions of x. □

To interpret the Bellman equation, recall that

$$rP = D + \dot{P},$$

can be interpreted as a relation where P is the present value of a payoff D with interest rate/discounting r (D is the dividends/return of an asset, $\dot{P}$ is capital gain/change in value). Here we hence see that, by definition, the value function is the present value of the period return, given a state x.

Bellman equation and the Maximum Principle

We now show the sense in which the Bellman equation and the first-order conditions above imply the equations for the Maximum Principle (i.e. Hamiltonian) derived previously.

Recall that in the optimal-control approach, $u^*(t)$ maximises

$$H(x, u, \lambda) = F(x, u) + \lambda g(x, c).$$

From the Bellman equation in the dynamic programming approach, $c^*(t)$ maximises

$$F(x, u) + V'(x) g(x, c) \tag{8}$$

Hence, the two approaches are consistent only if

$$\lambda \equiv V'(x);$$

i.e. the co-state in the calculus of variations approach is the derivative of the value function. This is consistent with the interpretation for the discounted value of the co-state variables offered before: the marginal value of an extra unit of the state variable.

Second, using that $\lambda \equiv V'(x)$, we can see that the first-order condition from optimal-control approach

$$H_c(x, c, \lambda) = 0$$

is equivalent to

$$F_c(x, u) + \lambda g_c(x, u) = 0,$$

which is the derivative of the Bellman equation with respect to c set to zero.

Third, moreover, differentiating the Bellman equation with respect to time yields

$$\begin{aligned}
\rho V'\dot{x} &= F_x\dot{x} + F_c u^{*\prime}\dot{x} + \left(V''g + V'g_x + V'g_c u^{*\prime}\right)\dot{x} \\
&= F_x\dot{x} + \left(V''g + V'g_x\right)\dot{x} + \Bigg(\underbrace{F_c + V'g_c}_{=0\because \text{FOC}}\Bigg) u^{*\prime}\dot{x} \\
&= F_x\dot{x} + \left(V''g + V'g_x\right)\dot{x}.
\end{aligned}$$

Outside of the steady state, $\dot{x} \neq 0$, so we can divide through to obtain

$$\rho V' = F_x + V''g + V'g_x \qquad (*)$$

Differentiating $\lambda \equiv V'(x)$ with respect to time yields and using that $\dot{x} = g(x, c)$,

$$\dot{\lambda} = V''\dot{x} = V''g.$$

Substituting above and $\lambda \equiv V'(x)$ into $(*)$ yields

$$\dot{\lambda} = \rho\lambda - \left(F_x + V'g_x\right),$$

which is equivalent to

$$\dot{\lambda} = \rho\lambda - H_x(x, c^{\star}, \lambda),$$

which is the law of motion of the co-state variable obtained using the Maximum Principle.

HJB with stochastic dynamics (diffusions)

We will extend this framework by introducing some Brownian risk to the dynamics of the states. Again the aim of the agent is to maximize its objective function:

$$v(t_0, X_{t_0}) = \sup_{\{c_t\}_{t_0}^T} \mathbb{E}_{t_0}\big(\int_{t_0}^T F(t, X_t, c_t)dt + \mathcal{F}(X_T)\big)$$

In the following we will consider the case of finite horizon, which becomes time-varying as we move closer to the date T. This case is used a lot in option pricing (where T is the maturity of the option). One could extend the situation to infinite horizon and add a transversality condition (but we would need to add strong assumption on the variance of diffusion). Here again v is the value function of the agent (at time t_0), F and $\mathcal{F}$ resp. the running gain and terminal gain. The agent controls c_t the (adapted) control variable and X_t is the state variable, which is the (unique) solution of Stochastic Differential Equation (SDE):

$$\begin{cases} dX_t = b(t, X_t, c_t)dt + \sigma(t, X_t, c_t)dB_t \\ X_{t_0} = x_0 \qquad\qquad (t_0, x_0) \in [0, T] \times \mathbb{R}^d \end{cases}$$

where $b(\cdot)$ is the drift, $\sigma(\cdot)$ the variance and B_t a Brownian motion. The Bellman equation is the following :

Proposition 7.3 (HJB with diffusion).

The Bellman equation is a Partial Differential Equation (PDE): the Hamilton-Jacobi-Equation (HJB), where we consider the case $x \in \mathbb{R}^M$. Two cases : the stationary case :

$$\rho v(x) = \sup_{c \in \Gamma(x)} \Big\{F(x, c) + \nabla_x v(x) \cdot b + \frac{1}{2}Tr\big(\sigma\sigma^T D_{xx}^2 v(x)\big)\Big\} = 0$$

and the finite-horizon, time varying case:

$$-\partial_t v(t, x) + \rho v(t, x) = \sup_{c \in \Gamma(x)} \Big\{F(t, x, c) + \nabla_x v(t, x) \cdot b + \frac{1}{2}Tr\big(\sigma\sigma^T D_{xx}^2 v(t, x)\big)\Big\}$$

Proof

To provide a general idea of the proof (of the time varying case), we start from a discrete time step, and take (carefully) the limit, to finally obtain a Partial Differential Equation (PDE): the Hamilton-Jacobi-Equation (HJB).

We start with the Bellman dynamic programming principle holds:

$$v(t_0, X_{t_0}) = \sup_{\{c_t\}_{t_0}^T} \mathbb{E}_{t_0}\big(\int_{t_0}^{t_1} e^{-\rho(t-t_0)} F(t, X_t, c_t)dt + e^{-\rho(t_1-t_0)} v(t_1, X_{t_1})\big)$$

The idea is to study "infinitesimal" variation in the value function. For that, we use the Itô formula to compute the value function at time $t + h$:

$$\sup_{\{c_t\}} \mathbb{E}_{t_0}\Big(\int_{t_0}^{t_0+h} F(t,x,c_t)dt + \int_{t_0}^{t_0+h} \Big\{\partial_t\, v - \rho\, v + \nabla_x\, v\cdot \boldsymbol{b}_t + \frac{1}{2}\,\mathrm{tr}\,\big(\sigma_t\sigma_t^T D^2_{xx}\, v\big)\Big\}dt + \int_{t_0}^{t_0+h} \nabla_x\, v\cdot \sigma_t\, d\mathbf{B}_t\Big) = 0$$

The expectation of stochastic integral is zero $\mathbb{E}(\int_0^t \cdots dB_s) = B_0 = 0$ by martingale property (an Ito stochastic integral always starts at value 0).

Take $h \to 0$ and applies the dominated convergence theorem, the integrand need to be zero for every t:

$$-\partial_t v(t,x) + \rho v(t,x) = \sup_c \Big\{ F(t,x,c) + \nabla_x v(t,x)\cdot\, b(t,x,c) + \frac{1}{2} Tr\big(\sigma(t,x,c)\sigma(t,x,a)^T D^2_{xx} v(t,x)\big)\Big\} = 0$$

This is the Hamilton Jacobi Bellman (HJB) PDE!

□

<u>*Note:*</u>

- Sometimes, mathematicians write the Hamilton-Jacobi-Bellman with "Hamiltonian"

$$H(t,x,p,M) = \sup_a \Big\{F(t,x,a) + p\cdot b + \frac{1}{2}Tr\big(\sigma\sigma^T M\big)\Big\}$$

  and the HJB rewrites:

$$-\partial_t v(t,x) + \rho v(t,x) = H(t,x,\nabla_x v, D^2_{xx} v) = 0$$

  The optimal control can be given in feedback form by the First-Order Conditions (FOC), which are:

$$F_c(t,x,c) + \nabla_x v(x)\cdot b_c(t,x,c) + \frac{1}{2}Tr\big(\sigma_c\sigma_c^T D^2_{xx} v(t,x)\big)$$

- Usual methods to find solutions lies in the verification methods: Guess a form for v analogous for the one in $F(x,c)$ and verify, c.f. the Merton example below.
- Question: What if the function v is not smooth?, i.e. not in $\mathcal{C}^{1,2}$ such that we can't apply the Ito's lemma.
  → Concept of viscosity solutions: Crandall and Lions (1989)

<u>*Example: RBC model in continuous time*</u>

Example 7.2.
The control problem of the household is

$$v(t_0,k_0,z_0) = \sup_{\{c_t\}_{t\geq 0}} \mathbb{E}_{t_0}\Big(\int_{t_0}^{\infty} e^{-\rho t} u(c_t)dt\Big)$$

with the diffusion dynamics for the productivity shocks z_t*:*

$$dk_t = (z_t F(k_t) - \delta k_t - c_t)dt$$
$$dz_t = \mu(z)dt + \sigma(z)dB_t$$

By applying the same methods, we can obtain the stationary HJB:

$$\rho v(k,z) = \max_c u(c) + \partial_k v(k,z)\big[zF(k) - \delta k - c\big] \\ + \mu(z)\partial_z v(k,z) + \frac{\sigma(z)^2}{2}\partial^2_{zz} v(k,z)$$

Example: Merton Consumption-investment problem

Example 7.3.

Consider an agent solving the following consumption-investment problem:

$$\begin{aligned} &\max \int_0^\infty e^{-\rho t} u(c_t) dt \\ s.t.\ &dn_t = n_t\Big(rdt + x_t((R-r)dt + \sigma dz_t)\Big) - c_t dt \\ &n_0 \text{ given} \end{aligned}$$

where c_t denotes consumption, x_t denotes the portfolio weight. The risk-free asset earns constant interest rate r and the risky asset has constant excess return $R - r$ and risk σ.

The HJB associated with the problem above is given by:

$$\rho V(n) = \max_{c,x} u(c) + V'(n)\Big(n(r + x(R-r)) - c\Big) + \frac{1}{2}V''(n)n^2x^2\sigma^2$$

Optimal policy is given by the F.O.C. and yields

$$\begin{cases} u'(c) &= V'(n) \\ V'(n)n(R-r) &= -V''(n)n^2x\sigma^2 \end{cases} \implies \quad x^\star(n) = x^\star = \frac{-V'(n)}{V''(n)n\sigma}\frac{(R-r)}{\sigma}$$

By Ito's lemma, $V'(n)$ follows:

$$dV'(n) = V''(n)dn + \frac{1}{2}V'''(n)d\langle n, n\rangle_t$$

Hence, $\frac{dV'(n)}{V'(n)} \equiv \frac{d\lambda}{\lambda}$ (the marginal value of state, i.e. the co-state) has volatility

$$\frac{V''(n)nx\sigma}{V'(n)} = -\frac{(R-r)}{\sigma}$$

Let $\xi_t \coloneqq e^{-\rho t}u'(c)$ be the *stochastic discount factor* process. Combining the result above and another application of Ito's formula, one can show that:

$$\frac{d\xi_t}{\xi_t} = -rdt - \frac{(R-r)}{\sigma}dB_t$$

Hence marginal utility of consumption has the same variance as marginal utility of wealth.

<u>*Special case*</u>

Proposition 7.4 (Value and Policy of Merton problem)**.**
If the utility has a CRRA form: $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$*, then the value function has the form:* $V(n) = \nu n^{1-\gamma}$ *and the constant* ν *can be solved from the HJB.*

More, it follows that the optimal policies are of the form:

$$
\begin{aligned}
c^\star(n) &= \left(\frac{\rho-(1-\gamma)r}{\gamma} - \frac{1}{2}\frac{(R-r)^2}{\sigma^2}\frac{1-\gamma}{\gamma^2}\right) n \\
s^\star(n) &= n(r+(R-r)x) - c = \left(\frac{r-\rho}{\gamma} + \frac{1+\gamma}{2\gamma}\frac{(R-r)^2}{\gamma\sigma^2}\right) a \\
x^\star(n) &= x^\star = \frac{R-r}{\gamma\sigma^2}
\end{aligned}
$$

In particular is the $u(c) = \log(c)$ *(i.e.* $\gamma = 1$*) then* $c^\star(n) = \rho n$ *and* $x^\star = \frac{R-r}{\gamma\sigma^2} = \frac{1}{\gamma\sigma}\times$ *Sharpe ratio (i.e. the risk adjusted excess return).*

<u>***Proof***</u>

Consider the problem

$$\rho v(n) = \max_{c,k} u(c) + v'(n)(rn + (R-r)k - c) + \frac{1}{2}v''(n)\sigma^2 n^2 x^2$$

Grouping terms by the relevant maximization problems and solving these, we can write

$$
\begin{aligned}
\rho v(n) &= H\left(v'(n)\right) + G\left(v'(n), v''(n)\right) + v'(n)ra \\
H(p) &= \max_c\{u(c) - pc\} = \frac{\gamma}{1-\gamma}p^{\frac{2-1}{\gamma}} \\
G(p,q) &= \max_x \left\{p(R-r)xn + \frac{1}{2}q\sigma^2 n^2 x^2\right\} = \frac{1}{2}\frac{p^2}{-q}\frac{(R-r)^2}{\sigma^2}
\end{aligned}
$$

and from the first-order conditions (FOC)

$$u'(c(n)) = v'(n), \quad x(n) = -\frac{v'(n)}{v''(n)}\frac{R-r}{\sigma^2}$$

Guess and verify $v(n) = \nu n^{1-\gamma}$ and hence $v'(n) = (1-\gamma)\nu n^{-\gamma}, v''(n) = -\gamma(1-\gamma)\nu n^{-\gamma-1}$

$$
\begin{aligned}
H\left(v'(n)\right) &= \frac{\gamma}{1-\gamma}\left(v'(n)\right)^{\frac{2-1}{\gamma}} = \frac{\gamma}{1-\gamma}((1-\gamma)\nu)^{\frac{2-1}{\gamma}} n^{1-\gamma} \\
\frac{(v'(n))^2}{-v''(n)} &= \frac{(1-\gamma)\nu}{\gamma}n^{1-\gamma} \\
G\left(v'(n), v''(n)\right) &= \frac{1}{2}\frac{(v'(n))^2}{-v''(n)}\frac{(R-r)^2}{\sigma^2} = \frac{1}{2}\frac{(R-r)^2}{\sigma^2}\frac{(1-\gamma)\nu}{\gamma}n^{1-\gamma}
\end{aligned}
$$

Substituting into the HJB equation and dividing by $\nu a^{1-\gamma}$, we have

$$\rho = \gamma((1-\gamma)\nu)^{-\frac{1}{\gamma}} + \frac{1}{2}\frac{(R-r)^2}{\sigma^2}\frac{1-\gamma}{\gamma} + (1-\gamma)r$$

From the formulas for the FOCs, we obtain : $c(n) = \gamma((1-\gamma)B)^{-\frac{1}{\gamma}}n$ and hence by replacing and rearranging the terms we obtain the three policy functions.

□