Problem Set – Macroeconomics model
Consumption and saving with risk

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Due Date: Monday 20th of September 2020

Description of this homework:

This homework should make sure that everyone knows and understand the basic models in Macroeconomics. It is aimed at recalling the central building block of the standard DSGE models[1]. This will be the foundation for the Neoclassical Growth model (Ramsey-Cass-Koopmans and its extension with risk by Brock-Mirman), the Real Business Cycle models (RBC) and New-Keynesian models (NK). These models will be covered extensively during the core macro sequence (Theory of Income). We will emphasize the market structure of the economy, which requires us to use Probability theory, stochastic processes and dynamic optimization (i.e. control theory) in discrete time.

This work is optional. No need to work hundreds of hours for this work. The main purpose is to make sure that all of you start with the same set of common knowledge. For that, there are more explanations included in the questions (to get you used to the notations/language).

Rules for submission:

This problem set should be handed in on the Canvas page on Monday, 20th, September 2020, before 6 pm Chicago-time. The solution will be posted online the day after the deadline. Problem sets can be handed in individually or in groups. (It is recommended to work in small groups).

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[1] DSGE of *Dynamic stochastic general equilibrium*: here the economic relations are dynamic (as agents are forward-looking) and stochastic (the economy is subject to probabilistic "shocks" and agents use rational expectations). Moreover, the equilibrium is "general" as the economy clears between aggregate demand (from a representative household) and aggregate supply (from a representative firm)
[However, for the sake of time, you will have to derive the household/consumer side of the model, and we'll leave the firm side for another homework]

0 Prerequisites, 10 points
Constrained optimization: Karush-Kuhn-Tucker Theorem

0.1 The static case

For an optimization problem, minimizing[2] a function $J(x)$ over a variable[3] x and subject to a set of M constraints $F_1(x) \leq 0 \ldots F_M(x) \leq 0$. Recall the (Karush-)Kuhn-Tucker (KKT) theorem that states necessary and sufficient conditions for optimality.

0.2 The dynamic case

We extend the problem in a dynamic environment, often met in economic models in discrete time. Now the objective function is time-separable and the budget constraints and the variables are time-specific. Therefore, there are an infinite number of constraints[4].

$$\min_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} J_t(x_t)$$
$$\text{s.t.} \qquad F_t(x_t, x_{t+1}) \leq 0 \qquad \forall t \geq 0$$

Adapt the previous KKT conditions to this new environment.

[2]Note, this is a minimization problem, minimizing "$J(x)$" is the "same" as maximizing "$-J(x)$". Formally: $\min_x J(x) = -\max_x -J(x)$

[3]Here, for simplicity, we suppose that $x \in \mathbb{R}$ or even $x \in \mathbb{R}^n$. The KKT theorem can be generalized to the case where $x \in X$ where X is a reflexive Banach space (therefore, x can be many different objects, for example a function! (in the case where X is an infinite-dimensional space)).

[4]Therefore, the Lagrange multipliers are time-specific: λ_t

1 The Household problem [Macro 1, 2 or 3], 20 points Deterministic setting and Lagrangian method

Now, let us consider the following Household problem:

$$\max_{\{C_t\}_0^\infty, \{A_{t+1}\}_0^\infty} \sum_{t=0}^{\infty} \beta^t U(C_t)$$
$$\text{s.t.} \qquad C_t + Q_t A_{t+1} \leq A_t + Y_t$$

A representative consumer receives income Y_t, which is time-varying (but that is deterministic, i.e. whose fluctuations can be perfectly anticipated in advance), consumes a good C_t, and save using an asset A. In period t, the consumer owns a quantity A_t of savings (when $A_t \leq 0$ this saving is in fact a debt) and decides how much to save for new period, but purchasing the asset at price Q_t.

Note that we could intuitively relabel this budget constraints as $C_t + B_{t+1} = (1 + r_t)B_t + Y_t$, with gross interest rate $(1 + r_t)$ and hence $Q_t = \frac{1}{1+r_t}$.

(a.) Apply the KKT theorem to this dynamic setting (i.e. use the first exercise/second case) to solve the Household problem. (Applying a theorem means checking explicitly the different hypothesis). The HH maximizes over its consumption path and savings path. You should obtain one optimality condition[5], which is called "Euler equation". This shows the relation between present and future marginal utility of consumption.

(b.) Apply these conditions to the special functional form of utility, CRRA in consumption $U(C_t) = \frac{C_t^{1-\sigma}-1}{1-\sigma}$, when σ is the risk aversion coefficient – which is also the *inverse* of intertemporal elasticity of substitution (IES).

- Explain this relation with economic intuitions.
- How the consumption path changes with Q_t? How do consumption and saving react when Q_t changes, in the case where IES is low ($\sigma > 1$) vs. when it is high ($\sigma < 1$).
- Lastly, when expressing Q_t as function of the other variables, how is the asset priced if $C_{t+1} \gg C_t$.

[5]Here, as there are 2 variables, there should be 2 types of optimality conditions, but here you should get rid of the Lagrange multipliers to determine explicitly the consumption/saving path. The reason is that one of the two "choice variable" is redundant/determined by the binding budget constraint.

2 Stochastic Income process [Macro, and Metrics 2 and 3], 40 points Random walk, Markov chain, Martingale, AR(1), e tutti quanti

We now introduce a simple shock for income $Y_t = Y(s_t)$. It fluctuate randomly given a stochastic process, where s_t is a "state of the world" (or alea) at date t that belongs to the information set $\mathcal{I}_t$ (what is known to be possible to happen at date t)[6]. More specifically, a sequence of state-of-the-world $\{s_0, s_1, \ldots, s_{t-1}, s_t\} = s^t$ defines a "history" s^t of shocks until t (this is the economist way of writing things compactly).

Putting some structure on this process $Y(s^t)$ (making it depends on past), we consider an Autoregressive – AR(1) – process for income:

$$\begin{cases} Y_t(s^t) & = \rho Y_{t-1}(s^{t-1}) + \varepsilon(s_t) \\ Y_{t_0} & = y \end{cases}$$

where $\rho \in \mathbb{R}$ is a parameter and ε is a i.i.d. random variable (i.e. a function of the state of the world), that simply[7] takes three values: $+1$ with proba p, -1 with proba q and 0 with proba $1 - p - q$. The following questions (b-f) are independent from each other (but most of them need the answer of (a.).)

(a.) Compute the conditional expectation $\mathbb{E}(Y_t(s^t)|\mathcal{I}_{t-1})$, as a function of the parameters and note that this expectation is a function of a variable depending on s^{t-1}.
- Using the same logic, write the value $\mathbb{P}(Y_t(s^t) = z|\mathcal{I}_{t-1})$ (Hint: you may need to use indicator functions $\mathbb{1}\{W = w\}$ for a random variables W and values w to be determined).

(b.) Show what are the conditions on the parameters (ρ, p and q, etc.[8]), for Y_t to be a martingale (or a sub-martingale, or a super-martingale).

(c.) Does this stochastic process respect the Markov property? Can we say that it is a Markov chain (i.e. Markov process)? What is the set of states? (Hint: this needs not be finite). Write the transition matrix as function of the parameters, for $\rho = 1$.

(d.) Assume that $\rho = 1$. This process is called a random walk. Suppose that there are many agents with the same income process and $y \in \mathbb{N}$. What is the law of motion of the distribution π_t over the income "states"?
- Give explicitly the distribution of income in date $t = 1$ and $t = 2$, when $y = 2$
- Do you expect this process to have a stationary distribution of over income states? (if yes, which one? if not, why not?)

(e.) Assume that $\rho < 1$. Write the expression of $Y(s^t)$ as a function of past shocks and parameters.
- In the VAR language, rewrite this with lag operators. Moreover, if the first date was $t_0 \to -\infty$, can you reexpress this process as a MA(∞) process? As a consequence, can you rewrite this lag polynomial in a more compact way? (Hint: try to "inverting" an operator "as if" it was a standard division and use the rule for geometric series).

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[6]In measure theory language, this information set is the $\sigma-$algebra generated by the stochastic process until t (and $\{\mathcal{I}_t\}_t$ is a filtration: $\mathcal{I}_{t-1} \subset \mathcal{I}_t \subset \mathcal{I}_{t+1}$), i.e. it implies that $Y(s_t)$ is measurable with respect to this $\sigma-$algebra: it doesn't include values that are "outside" this set (if measure theory is too abstract for you, don't worry and just ignore this last sentence/footnote!)

[7]If you find that too easy, feel free to consider that as a random variable $\varepsilon(s_t) \sim \mathcal{N}(\mu, \sigma)$

[8]This includes μ if you consider a Normal law for ε

- In this setting, (or in the case where $t_0 = 0$ and $y = 0$) can you compute the variance of $\mathbb{V}\text{ar}(Y_t)$ and the covariance $\mathbb{C}\text{ov}(Y_t, Y_{t-1})$? (Hint: recall that $\mathbb{V}\text{ar}(Z_t) = \mathbb{E}(Z_t^2) - \mathbb{E}(Z_t)^2$, for random variables Z_t that admit a finite second moment).
- Using the answer for the last subquestion, can you propose a method of moments (GMM) to estimate p, q and ρ.

(f.) Assume $\varepsilon \sim \mathcal{N}(\mu, \sigma)$ for this question. Suppose you had data on $Y_1, Y_2, \ldots, Y_T$, and you would like to estimate the parameters ρ, μ and σ using Maximum Likelihood estimation (MLE). Write the (log-)likelihood, i.e. the objective of this maximization problem.

(g.) Suppose ε is again the original three states $\{+1, 0, -1\}$ random variable, with probability p, q such that $\mathbb{E}(\varepsilon) = 0$ and $\mathbb{V}\text{ar}(\varepsilon) = 1$ (what are these values of p and q?. Assume that the process Y_t is both a martingale and Markov chain (i.e. we have the properties for questions (b.) or (c.)).
- What do you expect this process to look like when the time period becomes infinitesimally small, i.e. $\Delta t = t_{i+1} - t_i \to 0$. Can you give the name of this stochastic process?
- More formally, define the rescaled process

$$W^{(n)}(t) = \frac{Y_{\lfloor nt \rfloor}}{\sqrt{n}}, \qquad t \in [0, 1]$$

Can you determine if $W^{(n)}(1)$ converge in distribution toward a specific random variable? If yes, which one? What is the name of this theorem?
- (Advanced) Can one show this convergence for every t, and hence determine the law of $W(t)$ at the limit for n? What is the name of this theorem?

3 Household problem in AD economies [Macro 1, 3 / Price theory 2], 10 points State-dependent consumption & portfolio choice

Using a stochastic process for income, we assume that the market is complete in the Arrow Debreu (AD) sense: for each history s^t and each state of the world s_{t+1}, there exists an asset $A(s_{t+1}|s^t)$ that yield one unit of consumption in state s_{t+1}, that can be purchased at date s^t and at price $Q_t(s_{t+1}|s^t)$ (for compact notation: $A(s_{t+1}|s^t) \equiv A(s^{t+1})$ and $Q_t(s_{t+1}|s^t) = Q_t(s^{t+1})$). Now, given this complete market structure, the household/consumer maximizes its expected utility, given the probability $\pi(s^{t+1})$ of history s^{t+1}.

$$\max_{\{C_t(s^t)\}_0^\infty, \{A_{t+1}(s^{t+1})\}_0^\infty} \sum_{t=0}^{\infty} \sum_{s^t \subset \mathcal{I}_t} \beta^t \pi(s^t) U\big(C_t(s^t)\big)$$

$$\text{s.t.} \qquad C_t(s^t) + \sum_{s_{t+1}} Q_t(s_{t+1}|s^t) A(s_{t+1}|s^t) \leq A(s^t) + Y_t(s^t)$$

Similarly, using the Lagrangian optimization to derive the Euler equation, and specify the stochastic discount factor.

4 The Household problem with risk, [Macro 1, 2, 3], 20 points Self-insurance and Dynamic programming

Consider the same problem as in the previous exercise. This time, the market is incomplete : there does not exist assets for each states of the world s_{t+1}. The only asset is a "risk-free bond"

$$\max_{\{C_t(s^t)\}_0^\infty, \{A_{t+1}(s^{t+1})\}_{s_{t+1}}\}_0^\infty} \sum_{t=0}^{\infty} \sum_{s^t \subset \mathcal{I}_t} \pi(s^t)\beta^t U\big(C_t(s^t)\big)$$

$$\text{s.t.} \qquad C_t(s^t) + Q_t(s^t)A_{t+1}(s^t) \leq A_t(s^t) + Y_t(s^t)$$

To alleviate notations, you can remove the notations s^t where it is clear that the variables is *not* measurable w.r.t. future shocks, information $s_{t+1} \subset \mathcal{I}_{t+1}$ but $\not\subset \mathcal{I}_t$

For that, we will use a "Dynamic programming" procedure. Again, the HH maximizes over the consumption path and we will obtain a Euler equation.

The idea here is to solve[9] the "Bellman" equation: the utility function can be rewritten as an iterative "Value function":

$$\begin{aligned}
V_t(A_t, Y_t) &= \max_{\{C_t\}_t, \{A_{t+1}\}_t} \mathbb{E}\Big[\sum_{j=t}^{\infty} \beta^(j-t)U(C_j)\big|\mathcal{I}_t\Big] \\
&= \max_{\{C_t\}_t, \{A_{t+1}\}_t} \sum_{j=t}^{\infty} \sum_{s^j \subset \mathcal{I}_j} \pi(s^j|s^t)\beta^(j-t)U(C_j) \\
\Leftrightarrow \qquad V_t(A_t, Y_t) &= \max_{C_t, A_{t+1}} \Big[U(C_t) + \beta\mathbb{E}\big(V_{t+1}(A_{t+1}, Y_{t+1})\big|\mathcal{I}_t\big)\Big] \\
&= \max_{C_t, A_{t+1}} \Big[U(C_t) + \beta \sum_{s_{t+1} \subset \mathcal{I}_{t+1}} \pi(s_{t+1}|s^t)\, V_{t+1}(A_{t+1}, Y_{t+1}(s_{t+1}))\Big]
\end{aligned}$$

To solve this problem, you need to derive
(i) the KKT FOC-conditions for this new (iterative) maximization problem[10], and
(ii) a new "Envelope condition". This second condition shows the derivative of the value function (w.r.t. A_t) and, through the use of the chain-rule, is linked to the "Envelope Theorem". In this theorem, when a function is maximized over a variable (V_t is max over A_{t+1} here) and hence optimal in this sense, its derivative w.r.t. this same variable A_{t+1} is null.

Combining the different equations and manipulating the terms, show the Euler equation and show the similarities and differences with the Arrow-Debreu/complete market economy.

[9] During the lecture, we discussed the general method to solve such problem (and the proof that show that the sequential problem and its recursive formulation are equivalent.

[10] Note that we are back to the first "static" version of the KKT theorem (exo 0.1)

5 Numerical application : Monte Carlo and VFI

5.1 Part 1 – Stochastic processes, 20 points

Consider the stochastic process in part 2, in the random walk case :

$$\begin{cases} Y_t & = Y_{t-1} + \varepsilon \\ Y_{t_0} & = y \end{cases}$$

where ε is a i.i.d. random variable (i.e. a function of the state of the world), that simply takes three values: $+1$ with proba p, -1 with proba q and 0 with proba $1 - p - q$.

Random walk and Monte Carlo

1. Consider an alternative shock $\widetilde{\varepsilon} \sim \mathcal{N}(\mu, \sigma)$ such that $\widetilde{\varepsilon}$ has the same mean and variance as ε. What needs to be the relation between p, q on one side and μ and σ on the other? (part of this answer is already included in question 2.e. above I think).
2. Simulate $N = 5000$ samples of $T = 1000$ periods for the stochastic processes ε and Y_t starting at $y = 0$. (advice : don't try the full sample for N and T from the start, but start with small values to see if your code works, then change the N and T). For example take $p = 0.3$ and $q = 0.2$. Graph a plot with 10 paths of your simulations.
3. What should be the expectation and variance of Y_T given p and q? (use the question 1 here). Show a fine histogram of the distribution for Y_T (over N) against the density of a Normal distribution with the same theoretical mean and variance. Can we say that the Law of Large Number is a good approximation?
4. (A little harder) How would you use your simulations of ε in the question (2.) here to show that the process is a martingale/submartingale/supermartingale? More precisely, how would you compute $\mathbb{E}[Y_{t+1}|Y_t]$ using your 5000 samples of shocks ε_t?
5. (Optional/for cracks:) Implement a Doob-Meyer decomposition of Y_t.

Markov chains

1. Fill automatically a 200×200 Markov transition matrix from question (2.c) above, starting from states 0 to state 199. To ensure that the matrix rows still sum to ones, replace the terms that is "outside the matrix" at the boundary, by a "reflection": for example at 0 instead of falling to -1 with proba q and staying at 0 with proba $1 - p - q$, you replace it with a transition such that the chain stay with probability $1 - p$ (and rise with proba p of course). This is called a random walk with a "reflecting barrier" (here the barriers are -1 and 200), and this is used in some examples by F. Alvarez and R. Shimer.
2. Start from an initial distribution $\pi_0 = [1, 0, \dots, 0]$ as in the case above. Compute the law of motion of the distribution after $T = 1000$ iterations/periods.
3. Show a fine histogram of this distribution π_T for Y_T after $T = 1000$, against a Normal distribution with the same theoretical mean sample variance (a priori the same as in question 3. in the section "Random walk and Monte Carlo" here)

5.2 Part 2 – Dynamic Programming, 30 points

(Courtesy to Agustin Gutierrez for this exercise)

The objective of this exercise is to teach you how to solve recursive problems numerically. We work with the standard neoclassical growth model as we saw in class. Consider the standard neoclassical growth model:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln c_t$$

$$\text{s.t. } c_t + k_{t+1} = zk_t^{\alpha} + (1-\delta)k_t \forall t$$

$$c_t \geq 0 \qquad k_{t+1} \geq 0$$

$$k_0 \ \ given$$

Throughout this exercise, we consider the following configuration of parameters:

Table 1: Parameter Values

β	α	δ	z	k_0
0.98	1/3	1	1	$0.05k^{ss}$

The Theoretical Benchmark

You don't need to use the computer for this part. The objective is to solve the problem presented above so we can compare the answers we'll get with the computer. Note that the simple set-up we chose (log-preference plus full depreciation) is because we want closed form solutions.

1. Write the problem in recursive form (Bellman equation). What properties does the value function of this problem has?
2. Write down the first-order condition, the envelope condition and the Euler equation for this problem. Is the policy function $g(k)$ increasing or decreasing in the initial level of capital?
3. Find the steady state level of capital (Hint: consider the interior solution.)
4. Use the method of guess and verify to solve for $V(k)$ and the policy function $g(k)$. Guess that $V(k) = a\log(k) + b$ and verify. Plot the resulting policy and value functions. You need to set $\delta = 1$

The rest of the exercise will be done in the computer. The idea is to solve problem (1) numerically and compare our answers with those of this benchmark. The problem you are about to solve for does not require much computational power. However more complicated problem, if they are not coded properly, could take several hours to find the solution.

In order to start our computational trip we should discretize the state space of capital. We will approximate the function over the continuous domain by a discrete set of function values at a discrete set of points in the domain. Construct a grid of possible values of capital $\{k_1, k_2, \ldots, k_n\}$.

Set $k_1 = 0.05k^{ss}, k_n = 1.2k^{ss}$ and $n = 200$. Choose a relative error tolerance level, $\varepsilon = 1e^{-}5$

Value function iteration

Solve the problem using naive Value Function iteration (the algorithm is described below). Report the computational time that takes to solve it. Plot the resulting value and policy function (capital and consumption). How does your result compare with those in the theoretical benchmark?

Algorithm:

- *Step 1*: Start with an initial guess of the value function, $V^{(0)}(k)$. This is a vector of length n, i.e. $V^{(0)}(k) = \left\{V_i^{(0)}\right\}_{i=1}^{n}$, where $V_i^{(0)} = V^{(0)}(k_i)$. The initial guess could be whatever you want (Why?), in this case we set $V^{(0)}(k) = \mathbf{0}_{n\times 1}$
- *Step 2*: Update the value function using the Bellman equation. Specifically,

  (a) Fix the the current capital stock at one of the grid points, k_i, beginning with $i = 1$

  (b) For each possible choice of capital next period, $k_j, j = 1, \ldots, n$ calculate

  $$T_{ij} = u\left(zF\left(k_i, 1\right) + (1-\delta)k_i - k_j\right) + \beta V_j^{(0)}$$

  If consumption is negative or zero for a particular k_j, assign a large negative number to $T_{i,j}$. Note that T_i is a vector of length n, with each element T_{ij} representing the value of the maximize objective when starting with k_i we choose k_j.

  (c) Find the location of the maximum of T_i. Store the maximum as the i - th element of the updated value function $V^{(1)}(k)$. Store the location of the maximizer, as the i -the element in the policy vector g.

  (d) Choose a new grid point for the current capital stock in step (a) and repeat steps (b) and (c). Once we have completed steps (a) through (c) for each value of k_i we will have updated the value and policy functions and can continue to the next step.

- *Step 3*: Compute the distance between $V^{(0)}(k)$ and $V^{(1)}(k)$. A common definition is the sup norm, i.e.

  $$d = \max_{i\in\{1,2,\ldots,n\}} \left|V_i^{(1)} - V_i^{(0)}\right|$$

- *Step 4*: If the distance is within the tolerance error ε, the value function has converged so we have obtained the numerical estimates of the value and policy functions. If the distance is above ε return to Step 1 setting the initial guess to be the updated value function, i.e. $V^{(0)}(k) = V^{(1)}(k)$. Keep iterating until the value function has converged.

HINT: you should use at least 3 loops. A for-loop (or while-loop) for the convergence of V and two for-loops one for the initial capital level and the second one for the optimal choice of capital.

An improved in guess V_0

Repeat the previous exercise with $V_0\left(k_i\right) = \frac{u(f(k_i)-\delta k_i)}{1-\beta}$. Record the speed of convergence of the algorithm. What is the economic interpretation of this guess?