Problem Set – Macro 3

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Description of this homework:

This homework should make sure that everyone know and understand the seminal models in Macroeconomics. It is aimed at recalling the standard DSGE models[1] of Macroeconomics: Real Business Cycle models (RBC) and New-Keynesian models (NK). These models are covered in standard courses of Macro 2 - Business Cycle. For additional resources, see the slides of the course of J. Boehm (Sciences Po), or any standard textbook such as *D. Romer – Advanced Macroeconomics*.

This work will account for 10% of the grade of the course. No need to work hundreds of hours for this work. The main purpose is to make sure that all of you start with the same set of common knowledge.

Rules for submission:

Ideally, this problem set should be handed in on the Thursday 21st of September 2017, at the time of the TA session. However, 10 days may seem a bit short. We can decide a new deadline, for example: Monday 25th of September, if you feel the need to extend the deadline. The exact date will be discussed during the first TA session. Once the deadline is decided, no excuse nor delay will be accepted, as the solution will be posted online the day after the deadline. Problem sets should be handed in individually. However, it is strongly recommended to work in group for those who did not take the course in Sciences Po in first year.

[1] DSGE of *Dynamic stochastic general equilibrium*: here the economic relations are dynamic (as agents are forward-looking) and stochastic (the economy is subject to probabilistic "shocks" and agents use rational expectations). [However, for the sake of simplicity, you will have to derive these two models in a deterministic environment]. Moreover, the equilibrium is "general" as the economy clears between aggregate demand (from a representative household) and aggregate supply (from a representative firm)

1 Basics of constrained optimization: the Karush-Kuhn-Tucker Theorem (5 points)

1.1 The static case

For an optimization problem, minimizing[2] a function $J(x)$ over a variable[3] x and subject to a set of M constraints $F_1(x) \leq 0 \ldots F_M(x) \leq 0$. Recall the (Karush-)Kuhn-Tucker (KKT) theorem that states necessary and sufficient conditions for optimality.

Solution:

We suppose that the objective function J and the constraints $F_1, \ldots, F_M$ are continuous functions on the set X and derivable on the set K defined by the constraints $K = \{x \in X,\ \text{s.t. } F_i(x) \leq 0, \text{ for } 1 \leq i \leq M\}$. Introducing the "Lagrangian" $\mathcal{L}$ function associated to this problem:

$$\mathcal{L}(x, \lambda_1, \ldots, \lambda_M) = J(x) + \sum_{i=1}^{M} \lambda_i\, F_i(x) \qquad \forall (x, \lambda_i) \in X \times \mathbb{R}_+ \quad \forall 1 \leq i \leq M$$

The optimality condition is to find the solution x^* that reach a saddle point of this Lagrangian function, under the condition that the constraints are "qualified": *$\forall 1 \leq i \leq M$, the derivative of the constraint function $F_i'(u^*)$ should be negative (or equal to zero if F_i are affine). These conditions are sometimes called "Slater condition" in case of convex constraint functions, and "Mangasarian-Fromovitz constraint qualification" in the general case (where there are also equality constraint, which is not the case here). The main idea of qualification (very important for the proof of the "necessary condition" of KKT theorem) is that you can look in the neighborhood of the local minimum to find the optimality condition (after some "linearization" along the lines defined by gradients).*

Under all the previous hypothesis, the four following conditions are ***necessary*** for optimality. Formally, if x^* is a global minimum, then the four conditions are satisfied:

1. Stationarity:
$$J'(x^*) + \sum_{i=1}^{M} \lambda_i\, F_i'(x^*) = 0$$
(Equivalent to the "saddle point conditions" on the Lagrangian: $\frac{\partial \mathcal{L}}{\partial x}(x, \lambda) = 0$, $\frac{\partial \mathcal{L}}{\partial \lambda_i}(x, \lambda) = 0 \quad \forall\ 1 \leq i \leq M$)
2. Primal feasibility (simply, constraints should be satisfied): $\quad F_i(x^*) \leq 0 \quad \text{for} \quad 1 \leq i \leq M$
3. Dual feasibility: $\qquad \lambda_i \geq 0 \qquad \forall 1 \leq i \leq M$
4. Complementarity: $\qquad \sum_{i=1}^{M} \lambda_i\, F_i(x^*) = 0$
(If the constraint is binding at optimum ($F(x^*) = 0$) then the Lagrange multiplier is strictly positive (it stands for the "shadow value" of relaxing the constraint) and conversely)

[2]Note, this is a minimization problem, minimizing "$J(x)$" is the "same" as maximizing "$-J(x)$". Formally: $\min_x J(x) = -\max_x -J(x)$

[3]Here, for simplicity, we suppose that $x \in \mathbb{R}$ or even $x \in \mathbb{R}^n$. The KKT theorem can be generalized to the case where $x \in X$ where X is a reflexive Banach space (therefore, x can be many different objects, for example a function! (in the case where X is an infinite-dimensional space)).

Under an additional assumption, that the objective function J and the constraints $F_1, \dots, F_M$ are ***convex***, then these four conditions are also ***sufficient***.
Said differently, if x^* satisfy the four conditions, then x^* is global minimum.

1.2 The dynamic case

We extend the problem in a dynamic environment, often met in economic models in discrete time. Now the objective function is time-separable and the budget constraints and the variables are time-specific. Therefore, there are an infinite number of constraints[4].

$$\min_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} J_t(x_t)$$
$$\text{s.t.} \qquad F_t(x_t, x_{t+1}) \leq 0 \qquad \forall t \geq 0$$

Adapt the previous KKT conditions to this new environment.

Solution:

Assuming the regularity conditions stated above (continuity, derivability and qualification of the constraints), and assuming the convexity of objective and constraints functions, we can redefine the Lagrangian and restate the four (necessary and sufficient) optimality conditions:

$$\mathcal{L}(x_t, \lambda_t) = \sum_{t=0}^{\infty} J_t(x_t) \; + \; \sum_{t=0}^{\infty} \lambda_t \; F_t(x_t, x_{t+1})$$

1. Stationarity:
$$J_t'(x_t^*) + \lambda_t \, \frac{\partial F_t(x_t^*, x_{t+1})}{\partial x_t} + \lambda_{t-1} \, \frac{\partial F_{t-1}(x_{t-1}, x_t^*)}{\partial x_t} = 0, \qquad \forall \, t \geq 0$$
(Again equivalent to the "saddle point conditions" on the Lagrangian). Here, this condition should be satisfied at each point of time (Agents are forward-looking and the environment is deterministic here). Moreover, the fact that we end up with only $J_t'(x_t)$ is due to the time-separability of the objective function.
2. Primal feasibility: $\quad F_t(x_t^*, x_{t+1}) \leq 0 \quad \text{for} \quad t \geq 0$
3. Dual feasibility: $\quad \lambda_t \geq 0 \qquad \forall t \geq 0$
4. Complementarity: $\quad \sum_{t=0}^{\infty} \lambda_t \, F_t(x_t^*, x_{t+1}) = 0 \quad \forall t \geq 0$
(When the constraint is binding at time t, the Lagrange multiplier is strictly positive and stands for the "shadow value" of relaxing the constraint, i.e. the marginal amount you will be able to minimize on the objective function if you could loosen the inequality constraint a little bit))

[4]Therefore, the Lagrange multipliers are time-specific: λ_t

2 The Firm problem (3 points)

1) Suppose the representative firm should maximize its profit, in an fully-competitive environment, and adjusting its input factor (labor L_t and capital K_t). Profit are defined as output $Y_t = F_t(K_t, L_t)$ discounted for rental cost of capital ($R_t K_t$) and wages ($w_t L_t$). From this *static and unconstrained* problem, derive the First Order Condition, and show how marginal productivity of a factor equals its cost.

2) Suppose now that the technology shows constant return to scale, and in particular a Cobb-Douglas function: $F_t(K_t, L_t) = (K_t)^\alpha (A_t L_t)^{(1-\alpha)}$.

a) Show how the Cobb-Douglas technology has constant return to scale.

b) Plug the optimality condition in the objective function (the profits), and show that the profits are zero for this representative firm.

Solution:

1. This unconstrained problem takes the form:

$$\Pi_t(K_t, L_t) = F_t(K_t, L_t) - R_t K_t - w_t L_t$$
$$\max_{K_t, L_t} [Pi_t]$$

The first-order condition are the following:

$$\frac{\partial \Pi_t(K_t, L_t)}{\partial K_t} = \frac{\partial F(K_t, L_t)}{\partial K_t} - R_t = 0$$
$$\frac{\partial \Pi_t(K_t, L_t)}{\partial L_t} = \frac{\partial F(K_t, L_t)}{\partial L_t} - w_t = 0$$
$$\Rightarrow R_t = F_k(K_t, L_t)$$
$$\Rightarrow w_t = F_\ell(K_t, L_t)$$

2. Using a Cobb-Douglas function: $F(K_t, L_t) = (K_t)^\alpha (A_t L_t)^{(1-\alpha)}$, it has constant return to scale:

$$F_t(\theta K_t, \theta L_t) = (\theta K_t)^\alpha (A_t \theta L_t)^{(1-\alpha)} = \theta^{\alpha+1-\alpha} (K_t)^\alpha (A_t L_t)^{(1-\alpha)} = \theta \, F_t(K_t, L_t)$$

But, careful!, it has diminishing returns in each variables:

$$\frac{\partial F_t(K_t, L_t)}{\partial K_t} = \alpha \big(\tfrac{A_t L_t}{K_t}\big)^{1-\alpha} \qquad \text{decreasing in } K_t$$
$$\frac{\partial F_t(K_t, L_t)}{\partial L_t} = (1-\alpha) A_t^{1-\alpha} \left(\tfrac{L_t}{K_t}\right)^{-\alpha} \qquad \text{decreasing in } L_t$$

Using the FOC optimality conditions above, profits rewrites:

$$
\begin{aligned}
R_t &= \alpha\big(\tfrac{A_t L_t}{K_t}\big)^{1-\alpha} \\
w_t &= (1-\alpha)\big(\tfrac{A_t L_t}{K_t}\big)^{-\alpha} \\
\Pi_t(K_t, L_t) &= (K_t)^{\alpha}(A_t L_t)^{(1-\alpha)} - K_t\,\alpha\big(\tfrac{A_t L_t}{K_t}\big)^{1-\alpha} - L_t\,(1-\alpha)A_t^{1-\alpha}\,\big(\tfrac{L_t}{K_t}\big)^{-\alpha} \\
&= (K_t)^{\alpha}(A_t L_t)^{(1-\alpha)} - (\alpha + 1 - \alpha)(K_t)^{\alpha}(A_t L_t)^{(1-\alpha)} \\
&= 0
\end{aligned}
$$

This is a consequence of the assumption that technology has constant returns to scale.

3 The Household problem - Lagrangian method (3 points)

Now, let us consider the Household problem:

$$\max_{\{C_t\}_0^\infty, \{A_{t+1}\}_0^\infty, \{L_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$

$$\text{s.t.} \qquad C_t + A_{t+1} \leq (1 + r_t)A_t + w_t L_t$$

Apply the KKT theorem to this dynamic setting (i.e. use the first exercise/second case) to solve the Household problem. (Applying a theorem means checking explicitly the different hypothesis). The HH maximizes over its consumption path and labor (-supply) path.You should obtain (among other things) two[5] type of optimality conditions:

(a) an Euler equation showing relation between present and future marginal utility of consumption

(b) a Labor-Consumption tradeoff, with a ratio between marginal utility of consumption vs. marginal disutility of labor.

Apply these conditions to the special functional form of utility: separable between labor and consumption, CRRA in consumption and with a constant Frisch-elasticity: $U(C_t, L_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi}$ Explain briefly these relations in terms of economic mechanisms.

Solution:

We make the hypothesis that the utility function is continuous and derivable (expect when consumption is null, which is unlikely to be a optimum) and concave in consumption and labor. The constraint rewrites: $F_t(c, \ell, a_t, a_{t+1}) = C_t + A_{t+1} - (1 + r_t)A_t - w_t L_t \leq 0$. Note that A_{t+1} appears twice: once in F_t and once in F_{t+1} (recall the budget is a constraint at all time t). Therefore, the optimality of the path $(C_t^*, L_t^*)_{t=0}^\infty$ is equivalent to the following conditions:

1. Stationarity (C_t, A_{t+1} and L_t):

$$
\begin{array}{lll}
[C_t] & & \beta^t \dfrac{\partial U(C_t^*, L_t^*)}{\partial C_t} = \lambda_t \, (1), \qquad \forall\, t \geq 0 \\[2ex]
[A_{t+1}] & \{\dfrac{\partial U(C_t^*, L_t^*)}{\partial A_{t+1}} = 0\} & 0 = \lambda_t \, (1) + \lambda_{t+1}\{-(1 + r_{t+1})\} \\[2ex]
\text{(combining 1 \& 2)} & & \dfrac{\partial U(C_t^*, L_t^*)}{\partial C_t} = \beta \dfrac{\partial U(C_{t+1}^*, L_{t+1}^*)}{\partial C_t + 1} \beta(1 + r_{t+1}) \\[2ex]
[L_t] & & \beta^t \dfrac{\partial U(C_t^*, L_t^*)}{\partial L_t} = -\lambda_t \, w_t \\[2ex]
\text{(combining 1 \& 3)} & & \dfrac{\partial U(C_t^*, L_t^*)}{\partial L_t} = -\dfrac{\partial U(C_t^*, L_t^*)}{\partial C_t} \, w_t
\end{array}
$$

[5] Here, as there are 3 variables, there should be 3 types of optimality conditions, but here you should get rid of the Lagrange multipliers to determine explicitly the consumption path

Rewriting the previous equations more compactly:

$$U_c(C_{t+1})\beta(1+r_{t+1}) = U_c(C_t) \qquad [\textit{\textbf{Euler equation}}]$$

$$-\frac{U_\ell(C_t, L_t)}{U_c(C_t, L_t)} = w_t \qquad [\textit{\textbf{Labor consumption tradeoff}}]$$

2. Primal feasibility: $C_t + A_{t+1} \leq (1+r_t)A_t + w_t L_t \quad \text{for} \quad t \geq 0$

3. Dual feasibility: $\lambda_t \geq 0 \qquad \forall t \geq 0$

4. Complementarity: $\sum_{t=0}^{\infty} \lambda_t \big(C_t + A_{t+1} - (1+r_t)A_t + w_t L_t\big) = 0 \quad \forall t \geq 0$
   (the more the budget constraint binds, the more the Lagrange multiplier increases)

With the given functional form, the relations rewrite:

$$\beta(1+r_{t+1}) = \Big(\frac{C_{t+1}}{C_t}\Big)^{\sigma} \qquad\qquad C_t^{\sigma}\, L_t^{\varphi} = w_t$$

The Euler-equation[6] represent an intertemporal condition (everything else held constant, if future consumption increases or, if future interest rate increases, or if agents is less patient (smaller β), then the consumption today increases. Obviously, holding everything else constant is somehow counterintuitive: the Euler equation characterizes a "consumption smoothing" behavior: if the interest rate changes, the agent will adjust their saving considering substitution effect motives (interest rate rises, saving is more profitable and he will save more for the future), and income effect motives (rising interest rate means you will get the same (ex-post) saving amount with a smaller initial amount, therefore agents keep more money for present consumption and saving decreases).

The labor consumption tradeoff is a intra-temporal condition: everything else held constant, if labor increases (e.g. due to an exogenous rise in labor demand), then the consumption shall drop. In particular, this relation explains why the RBC model can difficultly generate positive/large government spending multipliers (mainly because higher government driven aggregate demand does not have any feedback effect on consumption behavior). Concerning real wages, the leisure-consumption trade-off can also be seen as a balance between substitution effect (if wage rise, and if leisure and consumption are substitute, people will exchange leisure against "cheaper" consumption) and income effect (need to work less to get the same consumption basket). RBC do not have the best foundations for labor supply, and that is why many recent papers include "search and matching" features in macro models.

[6] Which has not much to do with the swiss mathematician Euler ... only by the fact that Euler worked on optimality conditions in variational problems

4 The Household problem - Dynamic programming (4 points)

Now, let us consider the Household problem:

$$\max_{\{C_t\}_0^\infty, \{L_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$

$$\text{s.t.} \qquad C_t + A_{t+1} \leq (1+r_t)A_t + w_t L_t$$

Now, solve the Household problem using "Dynamic programming" method. Again, the HH maximizes over the consumption path and labor (supply) path and you should obtain the two type of optimality conditions:
(a) an Euler equation and (b) a Labor-Consumption tradeoff

The idea here is to solve[7] the "Bellman" equation: the utility function can be rewritten as an iterative "Value function":

$$V_t(A_t) = \max_{\{C_t\}_t, \{A_{t+1}\}_t, \{L_t\}_t} \Big[\sum_{j=t}^{\infty} \beta^j U(C_j, L_j) \Big]$$

$$\Leftrightarrow V_t(A_t) = \max_{C_t, A_{t+1}, L_t} \Big[U(C_t, L_t) + \beta V_{t+1}(A_{t+1}) \Big]$$

To solve this problem, you need to derive (i) the KKT FOC-conditions for this new (iterative) maximization problem[8], and (ii) a new "Envelope condition". This second condition shows the derivative of the value function (w.r.t. A_t) and, through the use of the chain-rule, is linked to the "Envelope Theorem". In this theorem[9], when a function is maximized over a variable (V_t is max over A_{t+1} here) and hence optimal in this sense, its derivative w.r.t. this same variable A_{t+1} is null.
Combining the different equations and manipulating the terms, you should obtain: (a) the Euler equation and (b) the Labor-consumption trade-off (exactly as before!).

Apply this conditions to the special functional form of utility: separable between labor and consumption, CRRA in consumption and with a constant Frisch-elasticity: $U(C_t, L_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi}$

[7] During the first TA session, we will discuss the general method to solve such problem (and the proof that show that the HH problem and its recursive formulation are equivalent.

[8] Note that we are back to the first "static" version of the KKT theorem (exo 1.1)

[9] We will talk about this during the first TA session as well

Solution:

As a preliminary exercise, let us recall the **Envelope Theorem**. This theorem arises when one want to differentiate a function that is expressed as an optimization problem. The most important case in economics (micro and macro) is when a value function depends on a "state-variable" x but is the optimum over a "control" variable y:

$$v(x) = \max_{y \in K} f(x, y) \qquad \forall x \in \mathbb{R}^d$$

We develop the multivariate case here, where: K is a compact set (eventually the set resulting from a set of constraints) $K \subset \mathbb{R}^m$ where m is the dimension of the control (y) and d the dimension of the state (x). We assume that, for every value of $y \in K$, $f(., y)$ is differentiable, and $\nabla_x f(x, y)$ is continuous.

We assume, after using the KKT theorem above if the case is appropriate, that this maximization problem has an unique optimum. This optimal y^* depends on the state variable: $y^*(x)$, implicitly derived from the KKT optimality conditions (and that can be analytically expressed using the ... implicit theorem!). In macro, the result of the maximization problem is often called the *policy function*: $y^*(x) = \pi(x)$.

Given all these assumptions, the value function v is of class C^1 and we can explicitly obtain its derivative:

$$\nabla v(x) = \nabla_x f(x, y^*(x)), \qquad \forall x \in \mathbb{R}^d$$

Important remark: the main thing here is that the derivative v is the same as the derivative of f w.r.t. its first variable, and $\nabla_y f(x, y^*(x)) = 0$. Why? Because the function $f(., y)$ is already optimal w.r.t. y and implied by the (KKT-) FOC conditions. Therefore, the chain rule implies ($d = m = 1$):
$v'(x) = f'_x(x, y(x)) + f'_y(x, y(x))\, y'(x) = f'_x(x, y(x))$

Alright? Ok, let's come back to our economic problem and our **Bellman equation**:

$$V_t(A_t) = \max_{C_t, A_{t+1}, L_t} \Big[U(C_t, L_t) + \beta V_{t+1}(A_{t+1})\Big]$$

Here, the value function depends on the state A_t, i.e. your saving, (in heterogeneous agents models, it will be the wealth), while it is optimal w.r.t. the controls C_t, A_{t+1}, L_t.

Therefore, we need to derive the **FOC conditions** of the problem at time t to obtain the policy functions ($C_t = \pi(A_t)$ and $L_t = \tilde{\pi}(A_t)$ both determined using the ... KKT theorem). Rewrite the inequality constraint and Lagrangian:

$F_t(c, \ell, a_{t+1}) = C_t + A_{t+1} - (1+r_t)A_t - w_t L_t \leq 0 \qquad \mathcal{L}_t(c, \ell, a_{t+1}, \lambda) = U(c, \ell) + \beta V_{t+1}(A_{t+1}) - \lambda_t\, F_t(c, \ell, a_{t+1})$

The FOC yields the following equations (note that we won't rewrite the primal and dual feasibility and complementarity conditions that are exactly the same as above) deriving w.r.t. C_t, A_{t+1} and L_t:

$$
\begin{array}{lll}
[C_t] & & \dfrac{\partial U(C_t^*, L_t^*)}{\partial C_t} = \lambda_t \ (1) \\
[A_{t+1}] & \{\dfrac{\partial U(c,\ell)}{\partial A_{t+1}} = 0\} & \beta \dfrac{dV_{t+1}(A_{t+1}^*)}{dA_{t+1}} = \lambda_t \ (1) \\
[L_t] & & \dfrac{\partial U(C_t^*, L_t^*)}{\partial L_t} = -\lambda_t \ w_t \\
\text{(combining 1 \& 3)} & & U_\ell(c^*, \ell^*) = -U_c(c^*, \ell^*) \ w_t \\
\text{(combining 1 \& 2)} & & U_c(c^*, \ell^*) = \beta V'_{t+1}(A_{t+1}^*)
\end{array}
$$

Here, the fourth equation is the ***Labor-consumption tradeoff*** (intratemporal FOC)! Now, to derive the **envelope condition**, we need to obtain the derivative of V_t (w.r.t. A_t) (why? In the fifth equation we still have the term $V'_{t+1}(A_{t+1})$ unknown). Coming back to the Bellman equation, let us obtain $V'(A_t)$. Here, we can "simply" use the Envelope theorem: we need to derive the value function w.r.t. the state variable, while it is already maximized over the three controls. ***But***, there is a trick, in the sense that, when the budget constraint is binding, one of the control can be expressed as a function of the other (state and control) variables:

$$C_t = \phi(A_t, A_{t+1}, L_t) = -A_{t+1} + (1 + r_t)A_t + w_t L_t$$

This implies that, after using the chain-rule, the envelope condition is expressed as:

$$V_t'(A_t) = \frac{\partial U(C_t, L_t)}{\partial C_t} \frac{\partial \phi(A_t, A_{t+1}, L_t)}{\partial A_t} + \beta \frac{dV_{t+1}(A_{t+1})}{dA_{t+1}} \frac{\partial A_{t+1}}{\partial A_t}$$

And ***here***, we can remember the Envelope theorem, telling us that this last term is zero, because V_t is already maximized over A_{t+1}, Therefore:

$$\begin{aligned} V_t'(A_t) &= \frac{\partial U(C_t, L_t)}{\partial C_t} \frac{\partial \phi(A_t, A_{t+1}, L_t)}{\partial A_t} \\ &= U_c(C_t, L_t) \, (1 + r_t) \end{aligned}$$

This condition holds at all time, and we can iterate ($t \rightarrow t+1$), and obtain:

$$V'_{t+1}(A_{t+1}) = U_c(C_{t+1}, L_{t+1}) \, (1 + r_{t+1})$$

and replacing in the fifth equation above :

$$U_c(C_t, L_t) = \beta V'_{t+1}(A_{t+1}) = \beta U_c(C_{t+1}, L_{t+1}) \, (1 + r_{t+1})$$

We got the ***Euler equation*** (intertemporal FOC)!

Again, we can recompute the two equations with a specific functional form:

$$\left(\frac{C_{t+1}}{C_t}\right)^{\sigma} = \beta(1 + r_{t+1}) \qquad\qquad C_t^{\sigma}\, L_t^{\varphi} = w_t$$

With the dynamic programming methods, that seems a lot of work for two simple (and already known) equations ... but, the good news is that the method is really general and can be applied to all kind of problems (especially when associated with the KKT theorem for inequality constraints). In this course, we will use the Dynamic programming approach to study Heterogenous agents models with incomplete markets and credit constraints. There will be two main methods: use the Bellman algorithm numerically (solving with an approximation of the value function in order to obtain the policy function) or using reduced heterogeneity (See Xavier Ragot's work for instance) to be able to derive the model analytically.

5 The New-Keynesian Model (2 points)

1. [Optional question] To solve the New Keynesian model, one of the step is to solve the following Household problem:

$$\max_{\{C_t\}_0^\infty, \{B_{t+1}\}_0^\infty, \{N_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$$

$$\text{s.t.} \qquad C_t P_t + Q_t B_{t+1} \leq B_t + W_t N_t + T_t$$

This problem is really similar to the one of the RBC problem (again everything is deterministic for the sake of simplicity). Using the given functional form $U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}$, derive the Euler-equation and labor-consumption tradeoff, using the method of your choice (the Lagrangian method or the Dynamic programming method)[10].

Briefly explain what are the changes in the Euler equation and Labor-consumption trade-off, for instance when inflation changes the price level.

2. ***[Compulsory question]*** Recall the log-linearized formulas[11] for (a) the dynamic IS-equation (linking the present and future output gaps and interest rate), (b) the New-Keynesian Phillips Curve (linking present and future inflation rates and aggregate demand) and (c) the interest rate rule[12] (showing how evolves the monetary policy). Describe these relations and their economic intuitions.

For the monetary policy rule, briefly explain the Taylor principle.

Solution:

1. In the New Keynesian models, the inter- and intra-temporal First Order Conditions rewrites:

$$Q_t = \beta\Big(\frac{P_t}{P_{t+1}}\Big)\Big(\frac{C_t}{C_{t+1}}\Big)^{\sigma} \qquad\qquad C_t^{\sigma} N_t^{\varphi} = \frac{W_t}{P_t}$$

Now, in the Euler equation, a wedge appears due to price level distortion: when prices are expected to increase in $t+1$ (again here there should be an expectation "$\mathbb{E}_t$" sign), then – everything else held constant – consumption (C_t) should increases: consumption smoothing is weighted given price level changes. Moreover, if wages decreases but if prices take time to adjust, then real wage fall and Household decrease their consumption and/or labor supply.

[10] This exercise is advised for those who feel the need to practice

[11] Simply states the three equations, no need to derive it! The main textbook for this type of model is: J. Gali - Monetary Policy, Inflation and the Business cycle

[12] There are different types of monetary policy rule, choose one and explain it

2. Noting that $i_t = -log(Q_t)$, the Euler equation described above yields, through its log-linearized version, the ***Dynamic IS equation***[13]:

$$\begin{aligned}
\hat{c}_t &= \mathbb{E}_t(\hat{c}_{t+1}) - \frac{1}{\sigma}(i_t - \mathbb{E}_t(\pi_{t+1}) - \rho) && \text{Euler equation} \\
\widetilde{y}_t &= \mathbb{E}_t(\widetilde{y}_{t+1}) - \frac{1}{\sigma}(i_t - \mathbb{E}_t(\pi_{t+1}) - r^n_t) && \text{IS equation} \\
\text{with } r^n_t &= \rho + \sigma\psi^n_{ya}\,\mathbb{E}_t(\Delta a_{t+1}) && \text{Natural int. rate}
\end{aligned}$$

Behind the IS equation lies the usual mechanisms underlying the standard Euler equations. However, the impact of monetary policy is clearer (it is a monetary model after all): when real interest rate rises above its natural level, this has a negative impact on consumption (as agents save instead of consuming), and output gap widens. Again, due to consumption smoothing motive, future consumption impacts linearly today's consumption: expectations of agents over future policy have therefore a critical impact (and this situation even creates what we call the "forward guidance" puzzle: in this type of model, future policy decision have oversized effects).

The ***New-Keynesian Phillips Curve*** is derived from the definition of marginal cost (directly coming from the labor-consumption tradeoff) and the price setting of firms:

$$\begin{aligned}
\hat{w}_t - p_t &= \sigma\hat{c}_t + \varphi\hat{n}_t && \text{labor-cons. tradeoff} \\
mc_t &= (\hat{w}_t - p_t) - mpn_t && \text{effective marginal cost} \\
\widehat{mc}_t &= \big(\sigma + \tfrac{\varphi+\alpha}{1-\alpha}\big)(y_t - y^n_t) = \psi_{mc}\,\widetilde{y}_t && \text{"marginal cost" gap} \\
\pi_t &= \beta\,\mathbb{E}_t(\pi_{t+1}) + \lambda\widehat{mc}_t && \text{firms price setting} \\
\pi_t &= \beta\,\mathbb{E}_t(\pi_{t+1}) + \kappa\,\widetilde{y}_t && \text{New Keynesian Philipps Curve}
\end{aligned}$$

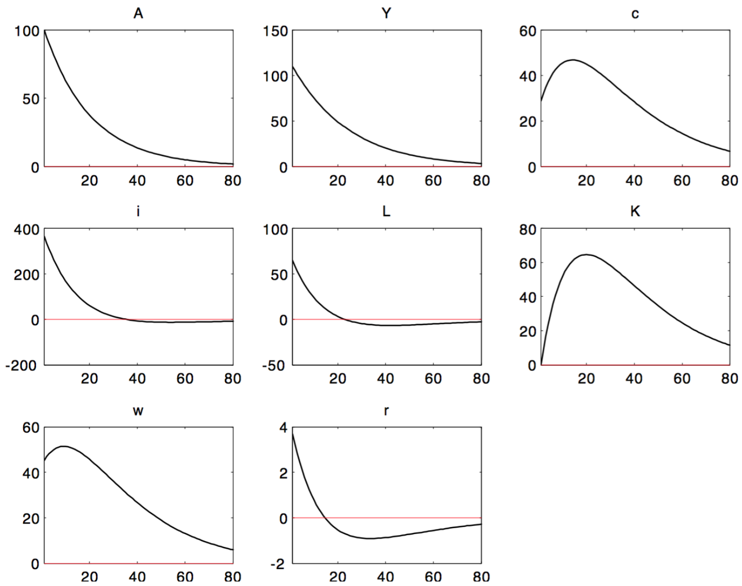
Behind the NKPC, the firms pricing decision have an impact on inflation: when aggregate demand increases, raising output above its flexible price level, it increases the labor disutility (households dislike working more than steady state level) and, through workers-firms bargaining, wages increases and so do marginal costs. However, only a fraction λ of firms update their prices (raise them). Moreover, when firms expect future inflation (and future rise in demand/marg. cost), they know that some of them will not be able to update their prices in the future (when the shocks arise) and prefer rising their prices now: this creates an expectation-driven "inflationary spiral".

[13]This time I provide the stochastic version, with expectation signs

6 The dynamics of the two models (3 points)

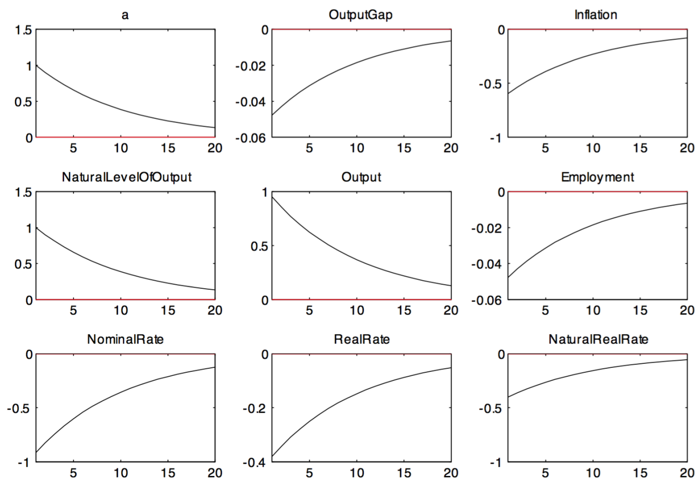
This is the response of the two standard models to a Technology shock:

Figure 1: *Baseline RBC model*



In the first RBC model, the variables are the following: A=TFP, Y=Output, C=consumption, i=Investment, L=Labor, K=Capital, w=Wages, r=interest rate. The relations structuring the models are the Euler equations, the Labor-consumption trade-off, the definition of output (technology: Cobb-Douglas, and $Y = C + i$), the evolution of capital ($K_{t+1} = (1-\delta)K_t + i$), the factor prices, wages and interest rate, as marginal productivity of factors, and an AR(1) evolution of TFP.

Figure 2: *Baseline NK model*



In the second NK model, the variables are described (with a=TFP), and the relations structuring the models are the IS equation, the NK-Phillips Curve, the monetary policy rule for the nominal interest rate, the definitions of: the natural rate of output and natural rate of interest (rates that would prevail in the flexible price economy), the output gap (effective output – natural output), the real rate as interest rate discounted for inflation, and employment.

Describe the economic mechanisms underlying the two sets of Impulse Response function (IRF) following a technology shock, first in the Real-Business Cycle model, and second in the New-Keynesian model.

Solution:

In the RBC model, when technology momentarily improves above trend, the marginal product of both capital and labor increases, which leads to higher wages and higher interest rates. Agents respond to the higher wage rate by increasing their labor supply. Output has to increase. Since agents want to have a smooth consumption pattern, aggregate consumption initially increases (proportionally) by less than output and, hence, investment increases (proportionally) by more than output. Capital slowly accumulates over time. The accumulation of capital lowers its marginal product and leads to an interest rate below its steady state level. Once technology is almost back to trend, there remains a wealth effect due to the higher capital stock. Agents therefore consume more while working slightly less than in steady state.

In the NK model, the favorable TFP shock increases the natural level of output (that would prevail under flexible prices), raises the natural interest rate (which is simply the discount rate adjusted for the evolution of technology) decreases firms marginal costs. However, prices are sticky and not all firms can adjust their prices (i.e. some firms would like to decrease them but can not because of Calvo-style staggered price setting). This relative price distortion reduces the output of those firms, the household consumption (due to monopolistic competition and "taste for diversity"), and therefore employment: output gap widens. Notice that the improvement in technology is partly accommodated by the monetary policy, easing nominal interest rate. Nevertheless, this expansionary policy is not sufficient to close the negative output gap, yielding deflation.