Problem Set 0 – Advanced Macroeconomics

Prerequisites: dynamic optimization, business cycles models
and consumption-saving problems

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Due Date: Monday, 14th of September 2026

Description of this homework:

This "Homework 0" should make sure that all of you start the course with the same set of tools and the same common background. It is aimed at recalling the central building blocks of the standard DSGE models[1] of modern macroeconomics: the Neoclassical Growth model (Ramsey–Cass–Koopmans and its stochastic extension by Brock–Mirman), the Real Business Cycle (RBC) model, the New-Keynesian (NK) model, and the consumption-saving problem with idiosyncratic risk (Bewley–Huggett–Aiyagari), which is the workhorse of the heterogeneous-agent literature. These models are covered in standard first-year sequences, and they will be the starting point of this course.

This problem set is organized in three blocks. ***Part 0*** recalls the mathematical toolbox – constrained optimization and the Karush–Kuhn–Tucker (KKT) theorem – that we will use throughout the course. ***Parts 1 to 5*** cover the representative-agent business-cycle models in a deterministic environment: the firm problem and the determination of factor prices, the household problem solved first with the Lagrangian method and then with dynamic programming, the New-Keynesian model, and the interpretation of impulse response functions in the two models. ***Parts 6 to 8*** introduce risk and market structure: stochastic processes for income, the household problem under complete markets in the Arrow–Debreu sense, and the incomplete-market "self-insurance" problem solved recursively. This last block requires probability theory, stochastic processes and dynamic optimization (i.e. control theory) in discrete time, which we will use extensively in the course.

This work will not be graded. No need to work hundreds of hours on it: the main purpose is that all of you start with the same set of common knowledge, and that you can identify by yourself the tools you may want to review. For that reason, more explanations than usual are included in the questions themselves (to get you used to the notations and to the language). The number of points indicated at the beginning of each part is only a rough indication of the time and effort that each of them requires. For additional resources, see any standard textbook, e.g. D. Romer, *Advanced Macroeconomics*, J. Galí, *Monetary Policy, Inflation and the Business Cycle* (for Parts 4–5), or L. Ljungqvist & T. Sargent, *Recursive Macroeconomic Theory* (for Parts 6–8).

Rules for submission:

This problem set should be handed in on Monday, 14th of September 2026 before the morning class. The exact date can still be discussed during the first class, if you feel the need to extend the deadline. Once the deadline is set, the solutions will be posted the day after, so no delay can be accepted. Problem sets should be handed in individually. However, it is strongly recommended to work in groups, especially for those of you who did not take the corresponding courses in the past years. Clarity and conciseness will be appreciated: for most questions, a few lines of derivation and two sentences of economic intuition are enough.

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[1] DSGE stands for *Dynamic Stochastic General Equilibrium*: the economic relations are dynamic (agents are forward-looking) and stochastic (the economy is subject to probabilistic "shocks" and agents form rational expectations). Moreover, the equilibrium is "general", as the economy clears between aggregate demand – from the households – and aggregate supply – from the firms. [For the sake of simplicity, the first parts are derived in a deterministic environment, and risk is introduced only from Part 6 onwards.]

0 Prerequisites (20 pts) Constrained optimization: Karush-Kuhn-Tucker Theorem

0.1 The static case

For an optimization problem, minimizing[2] a function $J(x)$ over a variable[3] x and subject to a set of M constraints $F_1(x) \leq 0 \dots F_M(x) \leq 0$. Recall the (Karush-)Kuhn-Tucker (KKT) theorem that states necessary and sufficient conditions for optimality.

0.2 The dynamic case in finite horizon

We extend the problem in a dynamic environment, often met in economic models in discrete time. Now the objective function is time-separable and the budget constraints and the variables are time-specific. Therefore, there are T constraints[4].

$$\begin{aligned} & \min_{\{x_t\}_{t=0}^{T}} \sum_{t=0}^{T} J_t(x_t) \\ \text{s.t.} \qquad F_t(x_t, x_{t+1}) \leq 0 \qquad & \forall 0 \geq t \geq T-1 \end{aligned}$$

Adapt the previous KKT conditions to this new environment.

0.3 The dynamic case in infinite horizon

TODO: include the infinite horizon case, and emphasis on the transversality condition to guaranty uniqueness.

[2] Note, this is a minimization problem, minimizing "$J(x)$" is the "same" as maximizing "$-J(x)$". Formally: $\min_x J(x) = -\max_x -J(x)$

[3] Here, for simplicity, we suppose that $x \in \mathbb{R}$ or even $x \in \mathbb{R}^n$. The KKT theorem can be generalized to the case where $x \in X$ where X is a reflexive Banach space (therefore, x can be many different objects, for example a function! (in the case where X is an infinite-dimensional space)).

[4] Therefore, the Lagrange multipliers are time-specific: λ_t

1 The Firm problem (10 pts) Profit maximization, factor prices, and constant returns to scale

1) Suppose the representative firm should maximize its profit, in an fully-competitive environment, and adjusting its input factor (labor L_t and capital K_t). Profit are defined as output $Y_t = F_t(K_t, L_t)$ discounted for rental cost of capital ($R_t K_t$) and wages ($w_t L_t$). From this *static and unconstrained* problem, derive the First Order Condition, and show how marginal productivity of a factor equals its cost.

2) Suppose now that the technology shows constant return to scale, and in particular a Cobb-Douglas function: $F_t(K_t, L_t) = (K_t)^{\alpha}(Z_t L_t)^{(1-\alpha)}$.
a) Show how the Cobb-Douglas technology has constant return to scale.
b) Plug the optimality condition in the objective function (the profits), and show that the profits are zero for this representative firm.

2 The Household problem (15 pts) Lagrangian method

Now, let us consider the Household problem:

$$\max_{\{C_t\}_0^\infty, \{A_{t+1}\}_0^\infty, \{L_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$
$$\text{s.t.} \qquad C_t + A_{t+1} \leq (1+r_t)A_t + w_t L_t$$

A representative consumer receives income $w_t L_t$, which is time-varying (but that is deterministic, i.e. whose fluctuations can be perfectly anticipated in advance), consumes a good C_t, and save using an asset A. In period t, the consumer owns a quantity A_t of savings (when $A_t \leq 0$ this saving is in fact a debt) and decides how much to save for new period A_{t+1}.

Note that we could intuitively relabel this budget constraints as $C_t + Q_t \tilde{A}_{t+1} = \tilde{A}_t + w_t L_t$, with the one-period asset $\tilde{A}_t = (1+r_t)A_t$ is bought at price $Q_t = \frac{1}{1+r_{t+1}}$, and "returns" 1 next period. This convention will be used in later sections.

(a.) Apply the KKT theorem to this dynamic setting (i.e. use the first exercise/second case) to solve the Household problem. (Applying a theorem means checking explicitly the different hypothesis). The HH maximizes over its consumption path and labor (-supply) path.You should obtain (among other things) two[5] type of optimality conditions:
(i) an Euler equation showing relation between present and future marginal utility of consumption
(ii) a Labor-Consumption tradeoff, with a ratio between marginal utility of consumption vs. marginal disutility of labor.
(b.) Apply these conditions to the special functional form of utility: separable between labor and consumption, CRRA in consumption and with a constant Frisch-elasticity: $U(C_t, L_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi}$, where σ is the risk aversion coefficient – which is also the *inverse* of intertemporal elasticity of substitution (IES), and φ is the Inverse of the Frish elasticity of labor supply.

[5] Here, as there are 3 variables, there should be 3 types of optimality conditions, but here you should get rid of the Lagrange multipliers to determine explicitly the consumption path

- Explain these relations with economic intuitions.
- How the consumption path changes with r_t? How do consumption and saving react when r_t changes, in the case where IES is low ($\sigma > 1$) vs. when it is high ($\sigma < 1$).
- Lastly, when expressing $Q_t = \frac{1}{1+r_{t+1}}$ as function of the other variables, how is the asset priced (what is Q_t), if $C_{t+1} \gg C_t$.

3 The Household problem, (15 pts) Dynamic programming

Now, let us consider the Household problem:

$$\max_{\{C_t\}_0^\infty,\{L_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$
$$\text{s.t.} \qquad C_t + A_{t+1} \leq (1+r_t)A_t + w_t L_t$$

This is exactly the same problem as in the previous section, but we now solve it ***recursively***, using the "Dynamic programming" method. Again, the HH maximizes over the consumption path and labor (supply) path, and you should obtain the very same two types of optimality conditions:
(i) an Euler equation and (ii) a Labor-Consumption tradeoff. The point of the exercise is the method, not the result: the two approaches must agree, and the recursive one is the only one that will still work once we introduce risk (in Section 8).

The idea here is to solve[6] the "Bellman" equation: the utility function can be rewritten as an iterative "Value function":

$$V_t(A_t) = \max_{\{C_t\}_t,\{A_{t+1}\}_t,\{L_t\}_t} \Big[\sum_{j=t}^{\infty} \beta^{j-t} U(C_j, L_j)\Big]$$
$$\Leftrightarrow V_t(A_t) = \max_{C_t, A_{t+1}, L_t} \Big[U(C_t, L_t) + \beta V_{t+1}(A_{t+1})\Big]$$

To solve this problem, you need to derive (i) the KKT FOC-conditions for this new (iterative) maximization problem[7], and (ii) a new "Envelope condition". This second condition shows the derivative of the value function (w.r.t. A_t) and, through the use of the chain-rule, is linked to the "Envelope Theorem". In this theorem[8], when a function is maximized over a variable (V_t is max over A_{t+1} here) and hence optimal in this sense, its derivative w.r.t. this same variable A_{t+1} is null.
Combining the different equations and manipulating the terms, you should obtain: (i) the Euler equation and (ii) the Labor-consumption trade-off (exactly as before!).

Apply this conditions to the special functional form of utility: CRRA in consumption and with a constant Frisch-elasticity: $U(C_t, L_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi}$

[6] During the course, we will discuss the general method to solve such problems (and prove that the sequential problem and its recursive formulation are equivalent).
[7] Note that we are back to the first "static" version of the KKT theorem (exo 0.1)
[8] We will come back to this theorem during the course as well

4 The New-Keynesian model, (10 pts) Nominal rigidities, IS-curve, Phillips curve and monetary policy

1. [Optional question] To solve the New Keynesian model, one of the step is to solve the following Household problem:

$$\max_{\{C_t\}_0^\infty,\{B_{t+1}\}_0^\infty,\{L_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$
$$\text{s.t.} \qquad C_t\, P_t + Q_t B_{t+1} \leq B_t + W_t L_t + T_t$$

This problem is really similar to the one of the RBC problem (again everything is deterministic for the sake of simplicity). Using the given functional form $U(C_t, L_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi}$, derive the Euler-equation and labor-consumption tradeoff, using the method of your choice (the Lagrangian method or the Dynamic programming method)[9].
Briefly explain what are the changes in the Euler equation and Labor-consumption trade-off, for instance when inflation changes the price level.

2. ***[Compulsory question]*** Recall the log-linearized formulas[10] for (a) the dynamic IS-equation (linking the present and future output gaps and interest rate), (b) the New-Keynesian Phillips Curve (linking present and future inflation rates and aggregate demand) and (c) the interest rate rule[11] (showing how evolves the monetary policy). Describe these relations and their economic intuitions.
For the monetary policy rule, briefly explain the Taylor principle.

[9] This exercise is advised for those who feel the need to practice

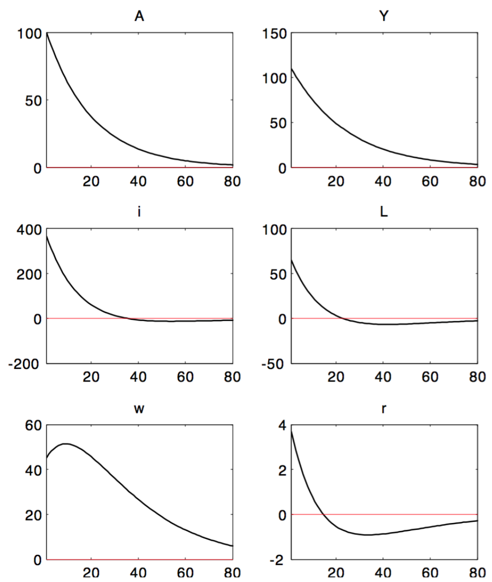
[10] Simply states the three equations, no need to derive it! The main textbook for this type of model is: J. Gali - Monetary Policy, Inflation and the Business cycle

[11] There are different types of monetary policy rule, choose one and explain it

5 The dynamics of the two models, (12 pts) Impulse response functions (IRF) to technology shock

This is the response of the two standard models to a Technology shock.[12]

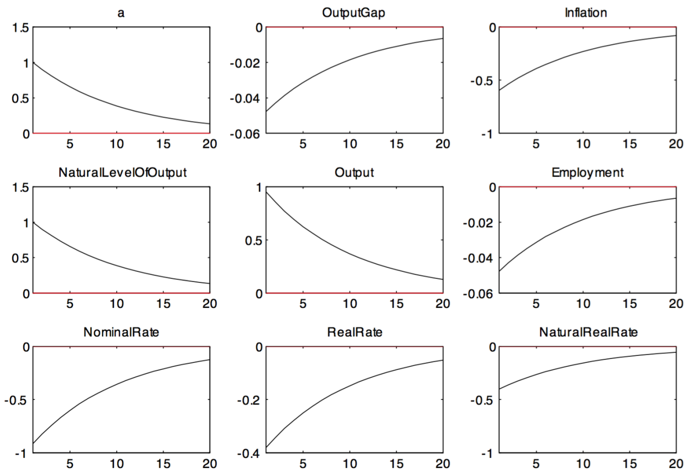
Figure 1: *Baseline RBC model*



In the first RBC model, the variables are the following: Z=TFP (labelled "A"), Y=Output, C=consumption, i=Investment, L=Labor, K=Capital, w=Wages, r=interest rate. The relations structuring the models are the Euler equations, the Labor-consumption trade-off, the definition of output (technology: Cobb-Douglas, and $Y = C + i$), the evolution of capital ($K_{t+1} = (1-\delta)K_t + i$), the factor prices, wages and interest rate, as marginal productivity of factors, and an AR(1) evolution of TFP.

[12] *Notation in the figures.* The plots were produced in Dynare/Matlab, with notations differing slightly from the ones used previously. In the *RBC* figure: "A" is the TFP Z_t, "Y" is output Y_t, "c" is consumption C_t, "i" is investment I_t, "L" is labor L_t, "K" is capital K_t, "w" is the real wage w_t and "r" is the interest rate r_t. In the *NK* figure: "a" is the (log) TFP z_t, "NaturalLevelOfOutput" is y_t^n, "Output" is y_t, "OutputGap" is the output gap $\widetilde{y}_t = y_t - y_t^n$, "Inflation" is π_t, "Employment" is $\hat{\ell}_t$, "NominalRate" is i_t, "RealRate" is the ex-ante real rate $i_t - \mathbb{E}_t(\pi_{t+1})$ and "NaturalRealRate" is r_t^n. In both figures, all variables are plotted as deviations from their steady-state value, and the horizontal axis is the number of periods after the shock.

Figure 2: *Baseline NK model*



In the second NK model, the variables are described (with z=TFP, labelled "a"), and the relations structuring the models are the IS equation (or Euler equation), the NK-Phillips Curve, the monetary policy rule for the nominal interest rate, the definitions of: the natural rate of output and natural rate of interest (rates that would prevail in the flexible price economy), the output gap (effective output – natural output), the real rate as interest rate discounted for inflation, and employment.

Describe the economic mechanisms underlying the two sets of Impulse Response function (IRF) following a technology shock, first in the Real-Business Cycle model, and second in the New-Keynesian model.

6 Stochastic income processes, (30 pts) Random walk, Markov chain, Martingale, AR(1), e tutti quanti

The first half of this problem set was written in *general equilibrium* and in a *deterministic* environment: households and firms interacted, and Y_t denoted aggregate output. In this second half, we zoom in on a single household in *partial equilibrium* – prices and the income process are taken as given – and we introduce *risk*. Accordingly, the notation shifts: from now on Y_t denotes the household's exogenous *income* ($\sim$ labor income $w_t L_t$, now random, and labor supply is no longer chosen by agents). This is the notation often used in the consumption-saving literature, the sections 7 and 8, and in the heterogeneous-agent literature more broadly.

We now introduce a simple shock for income $Y_t = Y(s_t)$. It fluctuate randomly given a stochastic process, where s_t is a "state of the world" (or alea) at date t that belongs to the information set $\mathcal{I}_t$ (what is known to be possible to happen at date t)[13]. More specifically, a sequence of state-of-the-world $\{s_0, s_1, \ldots, s_{t-1}, s_t\} = s^t$ defines a "history" s^t of shocks until t (this is the economist way of writing things compactly).

Putting some structure on this process $Y(s^t)$ (making it depends on past), we consider an Autoregressive – AR(1) – process for income:

$$\begin{cases} Y_t(s^t) & = \rho_Y Y_{t-1}(s^{t-1}) + \varepsilon(s_t) \\ Y_{t_0} & = y \end{cases}$$

where $\rho_Y \in \mathbb{R}$ is a parameter and ε is a i.i.d. random variable (i.e. a function of the state of the world), that simply[14] takes three values: $+1$ with proba p, -1 with proba q and 0 with proba $1-p-q$. The following questions (b-f) are independent from each other (but most of them need the answer of (a.).)

(a.) Compute the conditional expectation $\mathbb{E}(Y_t(s^t)|\mathcal{I}_{t-1})$, as a function of the parameters and note that this expectation is a function of a variable depending on s^{t-1}.
- Using the same logic, write the value $\mathbb{P}(Y_t(s^t) = z|\mathcal{I}_{t-1})$ (Hint: you may need to use indicator functions $\mathbb{1}\{W = w\}$ for a random variables W and values w to be determined).

(b.) Show what are the conditions on the parameters (ρ_Y, p and q, etc.[15]), for Y_t to be a martingale (or a sub-martingale, or a super-martingale).

(c.) Does this stochastic process respect the Markov property? Can we say that it is a Markov chain (i.e. Markov process)? What is the set of states? (Hint: this needs not be finite). Write the transition matrix as function of the parameters, for $\rho_Y = 1$.

(d.) Assume that $\rho_Y = 1$. This process is called a random walk. Suppose that there are many agents with the same income process and $y \in \mathbb{N}$. What is the law of motion of the

[13] In measure theory language, this information set is the $\sigma-$algebra generated by the stochastic process until t (and $\{\mathcal{I}_t\}_t$ is a filtration: $\mathcal{I}_{t-1} \subset \mathcal{I}_t \subset \mathcal{I}_{t+1}$), i.e. it implies that $Y(s_t)$ is measurable with respect to this $\sigma-$algebra: it doesn't include values that are "outside" this set (if measure theory is too abstract for you, don't worry and just ignore this last sentence/footnote!)

[14] If you find that too easy, feel free to consider that as a random variable $\varepsilon(s_t) \sim \mathcal{N}(\mu_Y, \sigma_Y)$

[15] This includes μ_Y if you consider a Normal law for ε

distribution π_t over the income "states"?
- Give explicitly the distribution of income in date $t = 1$ and $t = 2$, when $y = 2$
- Do you expect this process to have a stationary distribution of over income states? (if yes, which one? if not, why not?)

(e.) Assume that $\rho_Y < 1$. Write the expression of $Y(s^t)$ as a function of past shocks and parameters.
- In the VAR language, rewrite this with lag operators. Moreover, if the first date was $t_0 \to -\infty$, can you reexpress this process as a MA(∞) process? As a consequence, can you rewrite this lag polynomial in a more compact way? (Hint: try to "inverting" an operator "as if" it was a standard division and use the rule for geometric series).
- In this setting, (or in the case where $t_0 = 0$ and $y = 0$) can you compute the variance of $\mathbb{V}\mathrm{ar}(Y_t)$ and the covariance $\mathbb{C}\mathrm{ov}(Y_t, Y_{t-1})$? (Hint: recall that $\mathbb{V}\mathrm{ar}(Z_t) = \mathbb{E}(Z_t^2) - \mathbb{E}(Z_t)^2$, for random variables Z_t that admit a finite second moment).
- Using the answer for the last subquestion, can you propose a method of moments (GMM) to estimate p, q and ρ_Y.

(f.) Assume $\varepsilon \sim \mathcal{N}(\mu_Y, \sigma_Y)$ for this question. Suppose you had data on $Y_1, Y_2, \ldots, Y_T$, and you would like to estimate the parameters ρ_Y, μ_Y and σ_Y using Maximum Likelihood estimation (MLE). Write the (log-)likelihood, i.e. the objective of this maximization problem.

(g.) Suppose ε is again the original three states $\{+1, 0, -1\}$ random variable, with probability p, q such that $\mathbb{E}(\varepsilon) = 0$ and $\mathbb{V}\mathrm{ar}(\varepsilon) = 1$ (what are these values of p and q?. Assume that the process Y_t is both a martingale and Markov chain (i.e. we have the properties for questions (b.) or (c.)).
- What do you expect this process to look like when the time period becomes infinitesimally small, i.e. $\Delta t = t_{i+1} - t_i \to 0$. Can you give the name of this stochastic process?
- More formally, define the rescaled process

$$W^{(n)}(t) = \frac{Y_{\lfloor nt \rfloor}}{\sqrt{n}}, \qquad t \in [0, 1]$$

Can you determine if $W^{(n)}(1)$ converge in distribution toward a specific random variable? If yes, which one? What is the name of this theorem?
- (Advanced) Can one show this convergence for every t, and hence determine the law of $W(t)$ at the limit for n? What is the name of this theorem?

7 The Household problem in Arrow-Debreu economies, (10 pts) State-dependent consumption & portfolio choice

Using a stochastic process for income, we assume that the market is complete in the Arrow Debreu (AD) sense: for each history s^t and each state of the world s_{t+1}, there exists an asset $A(s_{t+1}|s^t)$ that yield one unit of consumption in state s_{t+1}, that can be purchased at date s^t and at price $Q_t(s_{t+1}|s^t)$ (for compact notation: $A(s_{t+1}|s^t) \equiv A(s^{t+1})$ and $Q_t(s_{t+1}|s^t) = Q_t(s^{t+1})$). Now, given this complete market structure, the household/consumer maximizes its expected utility, given the probability $\pi(s^{t+1})$ of history s^{t+1}.

$$\max_{\{C_t(s^t)\}_0^\infty, \{A_{t+1}(s^{t+1})\}_0^\infty} \sum_{t=0}^{\infty} \sum_{s^t \subset \mathcal{I}_t} \beta^t \pi(s^t) U\big(C_t(s^t)\big)$$

$$\text{s.t.} \qquad C_t(s^t) + \sum_{s_{t+1}} Q_t(s_{t+1}|s^t) A(s_{t+1}|s^t) \leq A(s^t) + Y_t(s^t)$$

Similarly, using the Lagrangian optimization to derive the Euler equation, and specify the stochastic discount factor.

8 The Household problem in incomplete markets (20 pts)

Consider the same problem as in the previous exercise. This time, the market is incomplete : there does not exist assets for each states of the world s_{t+1}. The only asset is a "risk-free bond"

$$\max_{\{C_t(s^t)\}_0^\infty, \{A_{t+1}(s^{t+1})\}_{s_{t+1}}\}_0^\infty} \sum_{t=0}^{\infty} \sum_{s^t \subset \mathcal{I}_t} \pi(s^t) \beta^t U\big(C_t(s^t)\big)$$

$$\text{s.t.} \qquad C_t(s^t) + Q_t(s^t) A_{t+1}(s^t) \leq A_t(s^t) + Y_t(s^t)$$

To alleviate notations, you can remove the notations s^t where it is clear that the variables is *not* measurable w.r.t. future shocks, information $s_{t+1} \subset \mathcal{I}_{t+1}$ but $\not\subset \mathcal{I}_t$

For that, we will use a "Dynamic programming" procedure. Again, the HH maximizes over the consumption path and we will obtain a Euler equation.

The idea here is to solve[16] the "Bellman" equation: the utility function can be rewritten as an iterative "Value function":

$$\begin{aligned} V_t(A_t, Y_t) &= \max_{\{C_t\}_t, \{A_{t+1}\}_t} \mathbb{E}\Big[\sum_{j=t}^{\infty} \beta^{j-t} U(C_j) \big| \mathcal{I}_t\Big] \\ &= \max_{\{C_t\}_t, \{A_{t+1}\}_t} \sum_{j=t}^{\infty} \sum_{s^j \subset \mathcal{I}_j} \pi(s^j | s^t) \beta^{j-t} U(C_j) \\ \Leftrightarrow \qquad V_t(A_t, Y_t) &= \max_{C_t, A_{t+1}} \Big[U(C_t) + \beta \mathbb{E}\big(V_{t+1}(A_{t+1}, Y_{t+1}) \big| \mathcal{I}_t\big)\Big] \\ &= \max_{C_t, A_{t+1}} \Big[U(C_t) + \beta \sum_{s_{t+1} \subset \mathcal{I}_{t+1}} \pi(s_{t+1}|s^t)\, V_{t+1}(A_{t+1}, Y_{t+1}(s_{t+1}))\Big] \end{aligned}$$

To solve this problem, you need to derive
(i) the KKT FOC-conditions for this new (iterative) maximization problem[17], and
(ii) a new "Envelope condition". This second condition shows the derivative of the value function (w.r.t. A_t) and, through the use of the chain-rule, is linked to the "Envelope Theorem". In this theorem, when a function is maximized over a variable (V_t is max over A_{t+1} here) and hence optimal in this sense, its derivative w.r.t. this same variable A_{t+1} is null.

Combining the different equations and manipulating the terms, show the Euler equation and show the similarities and differences with the Arrow-Debreu/complete market economy.

[16] During the lecture, we discussed the general method to solve such problem (and the proof that show that the sequential problem and its recursive formulation are equivalent.

[17] Note that we are back to the first "static" version of the KKT theorem (exo 0.1)