Problem Set 0 – Advanced Macroeconomics

Prerequisites: dynamic optimization, business cycles models and consumption-saving problems

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Due Date: Monday, 14th of September 2026

Description of this homework:

This "Homework 0" should make sure that all of you start the course with the same set of tools and background. It is aimed at recalling the central building blocks of the standard DSGE models[1] of modern macro: the Neoclassical Growth model (Ramsey–Cass–Koopmans and its stochastic extension by Brock–Mirman), the Real Business Cycle (RBC) model, the New-Keynesian (NK) model, and the consumption-saving problem with idiosyncratic risk (Bewley–Huggett–Aiyagari), which is the workhorse of the heterogeneous-agent literature.

This problem set is organized in three blocks. ***Part 0*** recalls the mathematical toolbox – constrained optimization and the Karush–Kuhn–Tucker (KKT) theorem – used throughout the course. ***Parts 1 to 5*** cover representative-agent business-cycle models in deterministic environments: the firm problem, the household problem solved first with both Lagrangian methods and dynamic programming, the New-Keynesian model, and the interpretation of IRFs in the two models. ***Parts 6 to 8*** introduce risk and market structure: stochastic processes for income, the household problem under complete markets with Arrow–Debreu securities, and incomplete-market "self-insurance" problem, both solved recursively. This last block requires probability theory, stochastic processes and dynamic optimization (i.e. control theory), which we will use extensively in the course.

This homework is *quite long*, so Section 6 is *optional*, as it relates more to macroeconometrics. In class, you will still be expected to be confortable with definitions/properties of stochastic processes, Markov Chains, AR(1) process, etc.

This work will not be graded. No need to work hundreds of hours on it: the main purpose is that all of you start with the same set of common knowledge, and that you can identify by yourself the tools you may want to review. For that reason, more explanations than usual are included in the questions themselves (to get you used to the notations and language). The number of points indicated at the beginning of each part is only a rough indication of the time and effort that each of them requires. For additional resources, see any standard textbook, e.g. D. Romer, *Advanced Macroeconomics*, J. Galí, *Monetary Policy, Inflation and the Business Cycle* (for Parts 4–5), or L. Ljungqvist & T. Sargent, *Recursive Macroeconomic Theory* (for Parts 6–8), or my Mathcamp lecture notes Part 1 and Part 2.
I will hold office-hours (3rd floor, EIEF main building) on Mon 07/09 and Wed 09/09, from 5 to 6 PM.

Rules for submission:

This problem set should be handed in on Monday, 14th of September 2026 before the morning class. Send an email to *thomas.bourany@eief.it* with the email subject "Macro Problem Set 0 [your name]". The solutions will be posted the day after, so no delay can be accepted. Problem sets should be handed in individually. However, it is strongly recommended to work in groups, especially for those of you who did not take the corresponding courses in the past years. Mention who you worked with at the top of your homework. Clarity and conciseness will be appreciated: for most questions, a few lines of derivation and 2-3 sentences of economic intuitions are enough.

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[1] DSGE stands for *Dynamic Stochastic General Equilibrium*: economic relations are dynamic (agents are forward-looking) and stochastic (the economy is subject to probabilistic "shocks" and agents form rational expectations). Moreover, the equilibrium is "general", as markets clear between aggregate demand – from households – and supply – from firms. [For simplicity, Parts 0-4 are derived in deterministic settings, and risk is introduced only from Part 6 onwards.]

0 Prerequisites (20 pts) Constrained optimization: Karush-Kuhn-Tucker Theorem

0.1 The static case

For an optimization problem, maximizing[2] a function $J(x)$ over a variable[3] x and subject to a set of M constraints $F_1(x) \leq 0 \dots F_M(x) \leq 0$. Recall the (Karush-)Kuhn-Tucker (KKT) theorem that states necessary and sufficient conditions for optimality.

Solution:

We suppose that the objective function J and the constraints $F_1, \dots, F_M$ are continuous functions on the set X and differentiable on the set K defined by the constraints $K = \{x \in X, \text{ s.t. } F_i(x) \leq 0, \text{ for } 1 \leq i \leq M\}$. Introducing the "Lagrangian" $\mathcal{L}$ function associated to this problem:

$$\mathcal{L}(x, \lambda_1, \dots, \lambda_M) = J(x) - \sum_{i=1}^{M} \lambda_i \, F_i(x) \qquad \forall (x, \lambda_i) \in X \times \mathbb{R}_+ \quad \forall 1 \leq i \leq M$$

The optimality condition is to find the solution x^* that reaches a saddle point of this Lagrangian function, under the condition that the constraints are "qualified"[4]:

Under all the previous hypotheses, the four following conditions are ***necessary*** for optimality. Formally, if x^* is a global maximum, then the four conditions are satisfied:

1. Stationarity:
$$J'(x^*) - \sum_{i=1}^{M} \lambda_i \, F_i'(x^*) = 0$$
(This is the "saddle point condition" on the Lagrangian in the direction of x: $\frac{\partial \mathcal{L}}{\partial x}(x^*, \lambda) = 0$. In the direction of the multipliers, the saddle point is *not* a zero derivative $\frac{\partial \mathcal{L}}{\partial \lambda_i} \geq 0$, as $\frac{\partial \mathcal{L}}{\partial \lambda_i} = -F_i(x)$ would force every constraint to bind, but a minimization over $\lambda \geq 0$, see what conditions 2. to 4. below express.)
2. Primal feasibility (simply, constraints should be satisfied): $F_i(x^*) \leq 0 \quad \text{for} \quad 1 \leq i \leq M$
3. Dual feasibility: $\lambda_i \geq 0 \qquad \forall 1 \leq i \leq M$
4. Complementarity: $\sum_{i=1}^{M} \lambda_i \, F_i(x^*) = 0$, which is equivalent to $\lambda_i \, F_i(x^*) = 0 \quad \forall 1 \leq i \leq M$
(If the constraint is slack at the optimum ($F_i(x^*) < 0$) then $\lambda_i = 0$: relaxing a constraint that does not bind is worthless. Conversely, $\lambda_i > 0$ requires the constraint to be binding, and λ_i is then its "shadow value". Careful: a binding constraint may still have $\lambda_i = 0$.)

Under an additional assumption, that the objective function J is ***concave*** and the constraints $F_1, \dots, F_M$ are ***convex***, then these four conditions are also ***sufficient***.
Said differently, if x^* satisfy the four conditions, then x^* is a global maximum.

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[2] Note, this is a maximization problem, maximizing "$J(x)$" is the "same" as minimizing "$-J(x)$". Formally: $\max_x J(x) = -\min_x -J(x)$

[3] Here, for simplicity, we suppose that $x \in \mathbb{R}$ or even $x \in \mathbb{R}^n$. The KKT theorem can be generalized to the case where $x \in X$ where X is a reflexive Banach space (therefore, x can be many different objects, for example a function! (in the case where X is an infinite-dimensional space)).

[4] The constraints are "qualified" when the feasible set near x^* is well approximated by the linearized constraints – this is what makes the KKT conditions *necessary*. Two standard sufficient conditions: the *Slater condition*, natural when the F_i are convex, requires a *strictly* feasible point, i.e. some $\bar{x}$ with $F_i(\bar{x}) < 0$ for every non-affine constraint (affine constraints only need to be satisfied); the *Mangasarian-Fromovitz constraint qualification*, for the general case, requires a direction d such that $\nabla F_i(x^*)' d < 0$ for every constraint *active* at x^*. In all the problems below the constraints are affine, so qualification holds automatically.

0.2 The dynamic case in finite horizon

We extend the problem in a dynamic environment, often met in economic models in discrete time. Now the objective function is time-separable and the budget constraints and the variables are time-specific. Therefore, there are $T+1$ constraints[5].

$$\max_{\{x_t\}_{t=1}^{T+1}} \sum_{t=0}^{T} J_t(x_t)$$

$$\text{s.t.} \qquad F_t(x_t, x_{t+1}) \leq 0 \qquad \forall\, 0 \leq t \leq T \qquad (x_0 \text{ given})$$

Adapt the previous KKT conditions to this new environment.

Solution:

Assuming the regularity conditions stated above (continuity, derivability and qualification of the constraints), and assuming the concavity of the objective and the convexity of the constraint functions, we can redefine the Lagrangian and restate the four (necessary and sufficient) optimality conditions:

$$\mathcal{L}(x_t, \lambda_t) = \sum_{t=0}^{T} J_t(x_t) \;-\; \sum_{t=0}^{T} \lambda_t \; F_t(x_t, x_{t+1})$$

1. Stationarity:
$$J_t'(x_t^*) - \lambda_t \, \frac{\partial F_t(x_t^*, x_{t+1})}{\partial x_t} - \lambda_{t-1} \, \frac{\partial F_{t-1}(x_{t-1}, x_t^*)}{\partial x_t} = 0, \qquad \forall\, 1 \leq t \leq T$$

   (Again equivalent to the "saddle point conditions" on the Lagrangian). Here, this condition should be satisfied at each point of time (Agents are forward-looking and the environment is deterministic here). Moreover, the fact that we end up with only $J_t'(x_t)$ is due to the time-separability of the objective function.

2. Primal feasibility: $F_t(x_t^*, x_{t+1}) \leq 0$ for $0 \leq t \leq T$

3. Dual feasibility: $\lambda_t \geq 0 \quad \forall\, 0 \leq t \leq T$

4. Complementarity: $\sum_{t=0}^{T} \lambda_t \, F_t(x_t^*, x_{t+1}) = 0$, equivalent to $\lambda_t \, F_t(x_t^*, x_{t+1}) = 0 \quad \forall\, 0 \leq t \leq T$

   (When the constraint is binding at time t, the Lagrange multiplier is strictly positive and stands for the "shadow value" of relaxing the constraint, i.e. the marginal amount you would gain on the objective function if you could loosen the inequality constraint a little bit)

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0.3 The dynamic case in infinite horizon

We now let the horizon go to infinity. The objective becomes an infinite sum with infinitely many constraints, and time-specific Lagrange multipliers λ_t. We introduce a discount factor $\beta \in (0,1)$: discounting is what makes the objective finite, hence the problem well posed, when J is bounded.[6]

$$\max_{\{x_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t J(x_t)$$

$$\text{s.t.} \qquad F_t(x_t, x_{t+1}) \leq 0 \qquad \forall\, t \geq 0 \qquad (x_0 \text{ given})$$

It would be tempting to take the conditions of Section 0.2 and simply replace "T" by "∞", but an infinite-horizon problem is *not* simply a finite-horizon problem with large T.

(a.) To the finite-horizon problem, add the restriction $x_{T+1} \geq 0$ with multiplier ν. Write the stationarity/complementarity slackness condition w.r.t. x_{T+1}. What terminal condition does it imply? (There

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[5] Therefore, the Lagrange multipliers are time-specific: λ_t

[6] On a balanced growth path with growth rate g and CRRA preferences with parameter σ, the objective is unbounded and the problem is well posed if $\beta(1+g)^{1-\sigma} < 1$.

you can assume $\partial F_t(x_t, x_{t+1})/\partial x_{t+1} = 1$ for simplicity for the rest of Section 0). Interpret it. What is the natural candidate for the analogue of this condition, when $T \to \infty$? what is it called? Explain intuitions.
(b.) Are the KKT conditions still necessary for optimality of the path $\{x_t\}_{t=0}^{\infty}$ when $T \to \infty$? Are they still sufficient for optimality of $\{x_t\}_{t=0}^{\infty}$?
(c.) Explain the nuances and differences between that transversality condition and the No Ponzi condition, which arise in the consumption-saving problem of Section 2 below, where $x_t \equiv A_t$ is the asset holding, $1 + r$ the gross return and $\lambda_t = \beta^t U'(C_t)$.

Solution:
(a.) In the finite-horizon problem, x_{T+1} appears in the last constraint F_T only, and yields no direct payoff. With the additional constraint $x_{T+1} \geq 0$ and its own multiplier $\nu \geq 0$, the First-Order Condition (FOC) with respect to x_{T+1}, together with the complementary slackness for $\nu \geq 0$, reads:

$$-\lambda_T \frac{\partial F_T(x_T, x_{T+1})}{\partial x_{T+1}} + \nu = 0 \qquad \text{together with} \qquad \nu \, x_{T+1} = 0$$

In the consumption-saving problems (with $x_t \equiv A_t$), we usually have $\partial F_T/\partial x_{T+1} = 1$, so that $\nu = \lambda_T > 0$: the multiplier λ_T on the last budget constraint is the shadow value of relaxing it, which is strictly positive. Complementary slackness then gives:

$$\lambda_T \, x_{T+1} = 0 \qquad \Longrightarrow \qquad x_{T+1} = 0$$

The interpretation is that "there is no value left on the table" when choosing x_{T+1} at the terminal date. When $T \to \infty$ there is no last period, and the natural analogue is to require that this "value left on the table" vanishes *asymptotically*:

$$\lim_{t \to \infty} \lambda_t \, x_{t+1} = 0$$

This is the ***transversality condition***. Note that it is *not* implied by the First-Order Conditions: it is an extra requirement, which is why it must be added to the optimality conditions.

(b.) In infinite horizon, the KKT conditions of Section 0.2 – the First-Order Conditions, the Primal and Dual Feasibility, and Complementary Slackness – remain ***necessary*** but are ***no longer sufficient***, even though J is concave and the constraint could be affine. This is the essential difference with the finite-horizon case: at T finite, backward induction from the terminal condition pins the path down. At $T \to \infty$, there is nothing to induct backwards from.

(c.) Finally, the two conditions – transversality and No-Ponzi – play very different roles. In the notation of the consumption-saving problem of Section 2:
- The ***no-Ponzi game*** condition $\lim_T \frac{A_{T+1}}{(1+r)^{T+1}} \geq 0$ is a *constraint imposed on the agent* by the market: nobody will lend to someone who plans to repay debt by issuing ever more debt. Without it, the problem has no solution – utility is unbounded, since any consumption path becomes affordable.
- The ***transversality condition*** $\lim_t \lambda_t A_{t+1} = \lim_t \beta^t U'(C_t) A_{t+1} = 0$ is a *property of the optimal path*: the agent does not *want* to leave wealth unused.
At the optimum, the agent saturates the no-Ponzi condition (leaving nothing behind), so the inequality holds with equality – which is why in practice the two are used interchangeably to pin down the path. Formally, for a concave problem, FOC + feasibility + transversality are *sufficient* for optimality.

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1 The Firm problem (10 pts) Profit maximization, factor prices, and constant returns to scale

1) Suppose the representative firm maximizes its profits, in a fully-competitive environment, adjusting its input factors (labor L_t and capital K_t). Output is $Y_t = Z_t F(K_t, L_t)$, where Z_t is the Total Factor Productivity (TFP), and profits are output net of the rental cost of capital ($R_t K_t$) and of wages ($w_t L_t$). From this *static and unconstrained* problem, derive the First Order Conditions, and show how the marginal productivity of a factor equals its cost.

2) Suppose now that the technology shows constant returns to scale, and in particular a Cobb-Douglas function: $F(K_t, L_t) = (K_t)^\alpha (L_t)^{1-\alpha}$, so that $Y_t = Z_t (K_t)^\alpha (L_t)^{1-\alpha}$.
a) Show how the Cobb-Douglas technology has constant returns to scale.
b) Plug the optimality conditions in the objective function (the profits), and show that the profits are zero for this representative firm.

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Solution:

1. This unconstrained problem takes the form:

$$\Pi_t(K_t, L_t) = Z_t F(K_t, L_t) - R_t K_t - w_t L_t$$
$$\max_{K_t, L_t} [\Pi_t]$$

The first-order conditions are the following:

$$\frac{\partial \Pi_t(K_t, L_t)}{\partial K_t} = Z_t \frac{\partial F(K_t, L_t)}{\partial K_t} - R_t = 0$$
$$\frac{\partial \Pi_t(K_t, L_t)}{\partial L_t} = Z_t \frac{\partial F(K_t, L_t)}{\partial L_t} - w_t = 0$$
$$\Rightarrow R_t = Z_t F_K(K_t, L_t)$$
$$\Rightarrow w_t = Z_t F_L(K_t, L_t)$$

2. Using a Cobb-Douglas function: $F(K_t, L_t) = (K_t)^\alpha (L_t)^{1-\alpha}$, it has constant returns to scale:

$$F(\theta K_t, \theta L_t) = (\theta K_t)^\alpha (\theta L_t)^{1-\alpha} = \theta^{\alpha+1-\alpha} (K_t)^\alpha (L_t)^{1-\alpha} = \theta\, F(K_t, L_t)$$

But, careful!, it has diminishing returns in each variable:

$$\frac{\partial Y_t}{\partial K_t} = \alpha\, Z_t \big(\tfrac{L_t}{K_t}\big)^{1-\alpha} \qquad \text{decreasing in } K_t$$
$$\frac{\partial Y_t}{\partial L_t} = (1-\alpha)\, Z_t \big(\tfrac{L_t}{K_t}\big)^{-\alpha} \qquad \text{decreasing in } L_t$$

Using the FOC optimality conditions above, profits rewrite:

$$\begin{aligned} R_t &= \alpha\, Z_t\big(\tfrac{L_t}{K_t}\big)^{1-\alpha} \\ w_t &= (1-\alpha)\, Z_t\big(\tfrac{L_t}{K_t}\big)^{-\alpha} \\ \Pi_t(K_t, L_t) &= Z_t(K_t)^{\alpha}(L_t)^{1-\alpha} - K_t\,\alpha\, Z_t\big(\tfrac{L_t}{K_t}\big)^{1-\alpha} - L_t\,(1-\alpha)\, Z_t\big(\tfrac{L_t}{K_t}\big)^{-\alpha} \\ &= Z_t(K_t)^{\alpha}(L_t)^{1-\alpha} - (\alpha + 1 - \alpha)\, Z_t(K_t)^{\alpha}(L_t)^{1-\alpha} \\ &= 0 \end{aligned}$$

This is a consequence of the assumption that technology has constant returns to scale.

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3) Take a step back to the firm's problem. Suppose output/value added Y is a general function of TFP, labor, and capital, i.e. $Y = ZF(L, K)$. Assuming perfect competition and constant returns to scale, derive the following expression for logged TFP: $z = y - (1-\alpha)\ell - \alpha k$, where lower-case letters denote logs of their level counterparts and $1-\alpha$ the wage bill as a share of total revenues (wL/Y).

Hint: Follow the procedure: (i) take the total derivative of $Y = f(Z, L, K) = ZF(L, K)$ w.r.t. each factor, (ii) divide by total quantity Y and multiply/divide by factor quantities in each quantity, (iii) use the assumption of perfect competition – where firms are minimizing costs and factor prices p_i of input q_i equal their marginal products $p_i = \frac{df}{dq_i}\frac{C}{Y}\mu$, where $\mu = \frac{\partial C}{\partial Y}\frac{Y}{C}$ is the cost elasticity of quantity produced (scale elasticity) – hence $\frac{df}{dq_i} = \frac{p_i}{\mu}\frac{Y}{C}$ (iv) use the assumption of constant returns to scale, where the scale elasticity $\mu = 1$, and (v) integrate the log differences $\int d\log x = \log x^*$, where the integration constant is chosen arbitrarily.

3.1) What does this approximation tell us about Cobb-Douglas production functions?
3.2) Is it a good way of estimating productivity? Are those assumptions realistic?

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Solution:
We follow the procedure to derive the variation in production as change in factors:

$$Y = f(Z, L, K) \qquad \Rightarrow \qquad dY = \frac{\partial f(\cdot)}{\partial Z}dZ + \frac{\partial f(\cdot)}{\partial K}dK + \frac{\partial f(\cdot)}{\partial L}dL$$

$$\frac{dY}{Y} = \frac{\partial f(\cdot)}{\partial Z}\frac{Z}{Y}\frac{dZ}{Z} + \frac{\partial f(\cdot)}{\partial K}\frac{K}{Y}\frac{dK}{K} + \frac{\partial f(\cdot)}{\partial L}\frac{L}{Y}\frac{dL}{L}$$

$$d\log Y = dy = \underbrace{\frac{\partial f(\cdot)}{\partial Z}\frac{Z}{Y}}_{=1}\underbrace{d\log Z}_{=dz} + \underbrace{\frac{\partial f(\cdot)}{\partial K}}_{=\frac{r}{\mu}\frac{Y}{C}}\frac{K}{Y}\underbrace{d\log K}_{=dk} + \underbrace{\frac{\partial f(\cdot)}{\partial L}}_{=\frac{w}{\mu}\frac{Y}{C}}\frac{L}{Y}\underbrace{d\log L}_{=d\ell}$$

$$dy = dz + \frac{1}{\mu}\frac{rK}{C}dk + \frac{1}{\mu}\frac{wL}{C}d\ell$$

The last step uses constant returns to scale *and* perfect competition twice: they give $\mu = 1$, and they make profits zero (question 2), i.e. total cost equals total revenue $C = Y$. The two cost shares rK/C and wL/C are therefore also *revenue* shares, $rK/Y = \alpha$ and $wL/Y = 1 - \alpha$:

$$\begin{aligned} & dy = dz + \alpha\, dk + (1-\alpha)\, d\ell \\ \Rightarrow \quad & y^* = z^* + \alpha k^* + (1-\alpha)\ell^* \qquad \Leftrightarrow \qquad z^* = y^* - \alpha k^* - (1-\alpha)\ell^* \end{aligned}$$

3.1) Nothing in the derivation assumed Cobb-Douglas. We used constant returns, cost minimization and competitive factor prices; the functional form appeared only at the integration step (v). The exponents are therefore the observed factor shares, not parameters of a technology, and any constant-returns technology looks Cobb-Douglas to a first order in logs. The approximation is exact only when

α is constant over the sample – and Cobb-Douglas is the production function for which it is. Hence the long-run stability of the labor share, one of Kaldor's (1961) facts, was read for decades as support for it; its decline since the 1980s (Karabarbounis and Neiman, 2014) can questioning these assumptions.

3.2) z is never observed. It is the Solow (1957) residual: what is left of output growth once measured inputs have been priced at their observed shares. Everything the assumptions get wrong therefore is captured in the TFP – which can therefore represent an "efficiency wedge" (Chari, Kehoe, McGrattan (2007)) instead of a pure technology parameter. Three cases matter in practice. Under imperfect competition price exceeds marginal cost, factor shares understate the output elasticities, and the residual moves with markups – Hall (1988) turns this around and uses the correlation to estimate them. Capital and labor enter as stocks while production uses their *services*: when utilization and effort vary over the cycle, measured z is procyclical with technology held fixed, which is most of what Basu, Fernald and Kimball (2006) correct for. And $\mu = 1$ is an assumption, not a measurement: with increasing returns the scale term never drops out. Growth accounting still runs on this residual, and the RBC models of Section 5 treat it as the technology shock – worth remembering when those models attribute TFP as a cause of business cycles.

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2 The Household problem (15 pts) Lagrangian method

Now, let us consider the Household problem:

$$\max_{\{C_t, A_{t+1}, L_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$

$$\text{s.t.} \qquad C_t + A_{t+1} \leq (1+r_t)A_t + w_t L_t \qquad \forall\, t \geq 0 \qquad (A_0 \text{ given})$$

$$\text{and} \qquad \lim_{T\to\infty} \frac{A_{T+1}}{\prod_{s=1}^{T+1}(1+r_s)} \geq 0$$

A representative consumer supplies labor L_t at the wage w_t, which is time-varying but deterministic (its fluctuations are perfectly anticipated), consumes a good C_t, and saves using an asset A. In period t, the consumer owns a quantity A_t of savings (when $A_t \leq 0$ this saving is in fact a debt) and decides how much to save for the new period, A_{t+1}. The second constraint is the no-Ponzi condition of Section 0.3, without which the problem has no solution.

Note that we could relabel this budget constraint as $C_t + Q_t \tilde{A}_{t+1} = \tilde{A}_t + w_t L_t$, where the one-period asset $\tilde{A}_{t+1}$ is bought at price $Q_t = \frac{1}{1+r_{t+1}}$ and "returns" 1 next period, with $\tilde{A}_t = (1+r_t)A_t$. This convention will be used in later sections.

(a.) Apply the KKT theorem of Section 0.3 to solve the Household problem (applying a theorem means checking explicitly its different hypotheses). The household maximizes over its consumption path, its saving path and its labor supply path. There are three choice variables, hence three first-order conditions; eliminating the Lagrange multipliers leaves two optimality conditions:
(i) an Euler equation, relating present and future marginal utility of consumption;
(ii) a labor-consumption tradeoff, relating the marginal utility of consumption to the marginal disutility of labor.
(b.) Apply these conditions to the special functional form of utility: separable between labor and consumption, CRRA in consumption and with a constant Frisch-elasticity: $U(C_t, L_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi}$, where σ is the risk aversion coefficient – which is also the *inverse* of the intertemporal elasticity of substitution (IES) – and φ is the inverse of the Frisch elasticity of labor supply.
- Explain these relations with economic intuitions.
- How does the consumption path change with r_t? How do consumption and saving react when r_t changes, in the case where the IES is low ($\sigma > 1$) vs. when it is high ($\sigma < 1$)?
- Lastly, expressing $Q_t = \frac{1}{1+r_{t+1}}$ as a function of the other variables, how is the asset priced (what is Q_t) if $C_{t+1} \gg C_t$?

Solution:
We assume that the utility function is continuous, differentiable (except when consumption is null, which is unlikely to be an optimum) and concave in consumption and labor. The budget constraint rewrites $F_t(C_t, L_t, A_t, A_{t+1}) = C_t + A_{t+1} - (1+r_t)A_t - w_t L_t \leq 0$; it is affine, so the constraints are qualified. Note that A_{t+1} appears twice: once in F_t and once in F_{t+1} (the budget is a constraint at every date t). Following Section 0, the Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t U(C_t, L_t) \; - \; \sum_{t=0}^{\infty} \lambda_t\, F_t(C_t, L_t, A_t, A_{t+1})$$

and the optimal path $\{C_t^*, L_t^*\}_{t=0}^\infty$ satisfies the following conditions:

1. Stationarity (w.r.t. C_t, A_{t+1} and L_t):

$$[C_t] \qquad \beta^t\, U_c(C_t^*, L_t^*) - \lambda_t = 0 \qquad \forall\, t \geq 0$$

$$[A_{t+1}] \qquad -\lambda_t + \lambda_{t+1}(1+r_{t+1}) = 0 \qquad \forall\, t \geq 0$$

$$[L_t] \qquad \beta^t\, U_\ell(C_t^*, L_t^*) + \lambda_t\, w_t = 0 \qquad \forall\, t \geq 0$$

Note that U does not depend on A_{t+1}, which is why the second line contains no utility term: assets are only a vehicle to move consumption across dates. Eliminating the multipliers between the first and second lines, and between the first and third, gives:

$$U_c(C_t, L_t) = \beta\,(1 + r_{t+1})\,U_c(C_{t+1}, L_{t+1}) \qquad [\textbf{\textit{Euler equation}}]$$
$$-\frac{U_\ell(C_t, L_t)}{U_c(C_t, L_t)} = w_t \qquad [\textbf{\textit{Labor-consumption tradeoff}}]$$

2. Primal feasibility: $F_t(C_t, L_t, A_t, A_{t+1}) = (C_t + A_{t+1} - (1 + r_t)A_t - w_t L_t) \leq 0$ for $t \geq 0$
3. Dual feasibility: $\lambda_t \geq 0 \quad \forall\, t \geq 0$
4. Complementarity: $\sum_{t=0}^{\infty} \lambda_t\, F_t(C_t, L_t, A_t, A_{t+1}) = \sum_{t=0}^{\infty} \lambda_t\,(C_t + A_{t+1} - (1+r_t)A_t - w_t L_t) = 0$ which is equivalent to $\lambda_t\, F_t = 0 \quad \forall\, t \geq 0$

Since U is strictly increasing in consumption, $\lambda_t > 0$ and the budget constraint binds at every date. Moreover, the Lagrange multiplier λ_t represents of the shadow value of relaxing the t-budget constraint. As shown in Section 0.3, these conditions (1.-4.) are necessary but not sufficient in infinite horizon.

5. Transversality condition $\lim_{t\to\infty} \beta^t U_c(C_t, L_t) A_{t+1} = 0$

This is what selects the optimal path among those satisfying the Euler equation. Therefore (1.-5.) (+concavity of $U(\cdot)$) are the sufficient conditions for optimality of the paths $\{C_t, L_t, A_{t+1}\}_{t\geq 0}$

With the given functional form, the two relations rewrite:

$$\beta(1 + r_{t+1}) = \Big(\frac{C_{t+1}}{C_t}\Big)^{\sigma} \qquad\qquad C_t^{\sigma}\, L_t^{\varphi} = w_t$$

or, equivalently, expressing the intertemporal condition in terms of the price $Q_t = \frac{1}{1+r_{t+1}}$ of the one-period asset:

$$Q_t \equiv \frac{1}{1 + r_{t+1}} = \beta\Big(\frac{C_t}{C_{t+1}}\Big)^{\sigma}$$
$$\text{or} \quad C_t = C_{t+1}\big(\beta(1 + r_{t+1})\big)^{-1/\sigma} = C_{t+1}\Big(\frac{\beta}{Q_t}\Big)^{-1/\sigma}$$

(b.) Let us provide intuitions.

The Euler equation[7] is an intertemporal condition. Everything else held constant, today's consumption C_t increases if future consumption C_{t+1} increases, or if the future interest rate r_{t+1} <u>decreases</u> – equivalently, if the asset price Q_t <u>rises</u> – or if the agent is less patient (smaller β).

Holding everything else constant is of course artificial: the Euler equation describes "consumption smoothing", balancing a substitution effect against an income effect. When the asset price Q_t falls, i.e. the interest rate r_{t+1} rises, the substitution effect makes saving more profitable and the agent saves more for future consumption. The income effect goes the other way: the same amount A_{t+1} can now be reached at a lower cost, so the agent keeps more for present consumption and gross saving falls. When $\sigma < 1$, i.e. with a high intertemporal elasticity of substitution (IES), the substitution effect dominates. When $\sigma > 1$, i.e. with a low IES, the income effect dominates.

This is also a pricing equation: Q_t is the value the agent puts on an asset that transfers wealth from t to $t + 1$. If $C_{t+1} \geq \beta^{1/\sigma} C_t$, then $Q_t \leq 1$ and the interest rate is positive. The asset price falls, and the interest rate rises, the larger the gap between future and present consumption, and the lower the IES (the higher σ). A low IES means a high value placed on transferring resources across dates.

The labor-consumption tradeoff is an intra-temporal condition: everything else held constant, if labor increases (e.g. because of an exogenous rise in labor demand), consumption must fall. It is also

[7] Which has not much to do with the swiss mathematician Euler ... only by the fact that Euler worked on optimality conditions in variational problems

why the RBC model struggles to generate large government spending multipliers: an increase in public spending works through the negative wealth effect it creates, which raises labor supply and output but crowds out private consumption and investment, so the multiplier stays below one. Concerning real wages, the leisure-consumption tradeoff again balances a substitution effect (if wages rise, and if leisure and consumption are substitutes, people exchange leisure for cheaper consumption) against an income effect (one needs to work less to afford the same consumption basket). RBC models do not have the best foundations for labor supply, which is why many recent papers add "search and matching" frictions.

In the Euler equation, once risk is introduced, the RHS $m_{t+1} = \beta U_c(C_{t+1}, \cdot)/U_c(C_t, \cdot)$ is what asset pricing calls stochastic discount factor (SDF) or pricing kernel (more on that in Sections 7 and 8).

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3 The Household problem (15 pts) Dynamic programming

Now, let us consider the Household problem:

$$\max_{\{C_t, A_{t+1}, L_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$
$$\text{s.t.} \qquad C_t + A_{t+1} \leq (1+r_t)A_t + w_t L_t \qquad \forall t \geq 0 \qquad (A_0 \text{ given})$$

This is exactly the same problem as in the previous section – with the same no-Ponzi condition – but we now solve it ***recursively***, using the "Dynamic programming" method. Again, the household maximizes over the consumption path and the labor supply path, and you should obtain the very same two types of optimality conditions:
(i) an Euler equation and (ii) a labor-consumption tradeoff. The point of the exercise is the method, not the result: the two approaches must agree, and the recursive one is the only one that will still work once we introduce risk (in Section 8).

The idea here is to solve[8] the "Bellman" equation: the objective can be rewritten as an iterative "Value function":

$$V_t(A_t) = \max_{\{C_j, A_{j+1}, L_j\}_{j \geq t}} \Big[\sum_{j=t}^{\infty} \beta^{j-t} U(C_j, L_j) \Big]$$
$$\Leftrightarrow V_t(A_t) = \max_{C_t, A_{t+1}, L_t} \Big[U(C_t, L_t) + \beta V_{t+1}(A_{t+1}) \Big]$$

To solve this problem, you need to derive (i) the KKT first-order conditions for this new (iterative) maximization problem[9], and (ii) a new "Envelope condition". This second condition gives the derivative of the value function (w.r.t. A_t) and, through the chain rule, is linked to the "Envelope Theorem". In this theorem[10], when a function is maximized over a variable (V_t is maximized over A_{t+1} here) and hence optimal in this sense, its derivative w.r.t. that same variable is null.
Combining the different equations and manipulating the terms, you should obtain (i) the Euler equation and (ii) the labor-consumption tradeoff (exactly as before!).

Apply these conditions to the special functional form of utility: CRRA in consumption and with a constant Frisch-elasticity: $U(C_t, L_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi}$

Solution:
As a preliminary exercise, let us recall the **Envelope Theorem**. This theorem arises when one wants to differentiate a function that is itself expressed as an optimization problem. The most important case in economics (micro and macro) is when a value function depends on a "state variable" x but is the optimum over a "control variable" y:

$$v(x) = \max_{y \in K} f(x, y) \qquad \forall x \in \mathbb{R}^d$$

We develop the multivariate case here, where: K is a compact set (eventually the set resulting from a set of constraints) $K \subset \mathbb{R}^m$ where m is the dimension of the control y and d the dimension of the state x. We assume that, for every value of $y \in K$, $f(., y)$ is differentiable, and $\nabla_x f(x, y)$ is continuous.

[8] During the course, we will discuss the general method to solve such problems (and prove that the sequential problem and its recursive formulation are equivalent).
[9] Note that we are back to the first "static" version of the KKT theorem (Section 0.1).
[10] We will come back to this theorem during the course as well.

We assume, after using the KKT theorem above if the case is appropriate, that this maximization problem has a unique optimum. This optimal y^* depends on the state variable: $y^*(x)$, implicitly derived from the KKT optimality conditions (and that can be analytically expressed using the ... implicit function theorem!). In macro, the result of the maximization problem is often called the *policy function.*

Given all these assumptions, the value function v is of class C^1 and we can explicitly obtain its derivative:
$$\nabla v(x) = \nabla_x f(x, y^*(x)), \qquad \forall x \in \mathbb{R}^d$$

Important remark: the main thing here is that the derivative of v is the same as the derivative of f w.r.t. its first variable, because $\nabla_y f(x, y^*(x)) = 0$. Why? Because $f(x, \cdot)$ is already optimal w.r.t. y, by the (KKT-) FOC conditions. Therefore, the chain rule implies ($d = m = 1$):
$v'(x) = f'_x(x, y(x)) + f'_y(x, y(x))\, y'(x) = f'_x(x, y(x))$

Let us come back to our economic problem and our **Bellman equation**:

$$V_t(A_t) = \max_{C_t, A_{t+1}, L_t} \Big[U(C_t, L_t) + \beta V_{t+1}(A_{t+1})\Big]$$

Here, the value function depends on the state A_t, i.e. your saving (in heterogeneous agents models, it will be the wealth), while it is optimal w.r.t. the controls C_t, A_{t+1}, L_t.

We first derive the **first-order conditions** of the problem at date t, which give the policy functions $C_t = \mathcal{C}(A_t)$, $L_t = \mathcal{H}(A_t)$ and $A_{t+1} = \mathcal{A}(A_t)$, all determined using the ... KKT theorem. The constraint and the Lagrangian are:

$$F_t(C_t, L_t, A_t, A_{t+1}) = C_t + A_{t+1} - (1 + r_t)A_t - w_t L_t \leq 0$$

$$\mathcal{L}_t(C_t, L_t, A_{t+1}, \lambda_t \big| A_t) = U(C_t, L_t) + \beta V_{t+1}(A_{t+1}) - \lambda_t\, F_t(C_t, L_t, A_t, A_{t+1})$$

Deriving w.r.t. C_t, A_{t+1} and L_t yields the following conditions (the primal and dual feasibility and the complementarity conditions are exactly the same as in the previous section, and we do not rewrite them):

$$\begin{array}{ll}
[C_t] & U_c(C_t^*, L_t^*) - \lambda_t = 0 \\
[A_{t+1}] & \beta\, V'_{t+1}(A_{t+1}^*) - \lambda_t = 0 \\
[L_t] & U_\ell(C_t^*, L_t^*) + \lambda_t\, w_t = 0 \\
\text{(combining 1 \& 3)} & U_\ell(C_t^*, L_t^*) = -U_c(C_t^*, L_t^*)\, w_t \\
\text{(combining 1 \& 2)} & U_c(C_t^*, L_t^*) = \beta\, V'_{t+1}(A_{t+1}^*)
\end{array}$$

Note that the multiplier λ_t is no longer discounted by β^t: the Bellman equation is written from the point of view of date t, so λ_t is the marginal value of relaxing the date-t budget constraint, in date-t utility. The fourth line is the ***labor-consumption tradeoff*** (the intratemporal condition), and we recover it directly. The last line is not yet the Euler equation: it still contains the unknown $V'_{t+1}(A_{t+1})$. To obtain it, we need the **envelope condition**, i.e. the derivative of V_t w.r.t. the state A_t. Here we can use the Envelope theorem, since V_t is already maximized over the three controls. ***But*** there is a trick: when the budget constraint binds, one of the controls can be expressed as a function of the state and of the other controls,

$$C_t = \phi(A_t, A_{t+1}, L_t) = -A_{t+1} + (1 + r_t)A_t + w_t L_t$$

so that the value function becomes an unconstrained maximum over (A_{t+1}, L_t). The chain rule then gives:
$$V'_t(A_t) = U_c\, \frac{\partial \phi}{\partial A_t} + \underbrace{\Big[U_c\, \frac{\partial \phi}{\partial A_{t+1}} + \beta\, V'_{t+1}(A_{t+1})\Big]}_{=0 \ \text{(FOC on } A_{t+1})} \frac{dA_{t+1}}{dA_t} + \underbrace{\Big[U_c\, \frac{\partial \phi}{\partial L_t} + U_\ell\Big]}_{=0 \ \text{(FOC on } L_t)} \frac{dL_t}{dA_t}$$

The two brackets are exactly the first-order conditions with respect to the controls A_{t+1} and L_t – recall $\partial\phi/\partial A_{t+1} = -1$ and $\partial\phi/\partial L_t = w_t$ – hence they are zero at the optimum. Note that it is the *whole bracket* that vanishes, not $\beta V'_{t+1}$ alone, which equals $U_c \neq 0$. Therefore:

$$
\begin{aligned}
V'_t(A_t) &= U_c(C_t, L_t)\,\frac{\partial\phi(A_t, A_{t+1}, L_t)}{\partial A_t} \\
&= U_c(C_t, L_t)\,(1 + r_t)
\end{aligned}
$$

This condition holds at all dates, so we can iterate it forward ($t \to t+1$) and obtain:

$$V'_{t+1}(A_{t+1}) = U_c(C_{t+1}, L_{t+1})\,(1 + r_{t+1})$$

Replacing in the last first-order condition above:

$$U_c(C_t, L_t) = \beta\,V'_{t+1}(A_{t+1}) = \beta\,U_c(C_{t+1}, L_{t+1})\,(1 + r_{t+1})$$

We recover the ***Euler equation*** (the intertemporal condition), identical to the one obtained with the Lagrangian in the previous section. As there, it has to be completed by the transversality condition $\lim_{t\to\infty} \beta^t U_c(C_t, L_t) A_{t+1} = 0$ to select the optimal path.

Again, we can recompute the two equations with a specific functional form:

$$\left(\frac{C_{t+1}}{C_t}\right)^{\sigma} = \beta(1 + r_{t+1}) \qquad\qquad C_t^{\sigma}\,L_t^{\varphi} = w_t$$

With the dynamic programming method, that seems a lot of work for two simple (and already known) equations ... but the good news is that the method is completely general: we will re-use it in Section 8, where risk makes the sequential Lagrangian approach much less convenient.

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4 The New-Keynesian model (10 pts) Nominal rigidities, IS-curve, Phillips curve and monetary policy

1. [Optional question] To solve the New Keynesian model, one of the steps is to solve the following Household problem:

$$\max_{\{C_t, B_{t+1}, L_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$
$$\text{s.t.} \qquad C_t\, P_t + Q_t B_{t+1} \leq B_t + W_t L_t + T_t$$

Here P_t is the price level, W_t the nominal wage, B_{t+1} a one-period nominal bond bought at price Q_t and paying one unit of currency at $t+1$, and T_t lump-sum transfers (including firms' profits). This problem is really similar to the one of Section 2 (again everything is deterministic for the sake of simplicity). Using the given functional form $U(C_t, L_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi}$, derive [or simply state] the Euler equation and the labor-consumption tradeoff, using the method of your choice (the Lagrangian method or the Dynamic programming method)[11].
Briefly explain what changes in the Euler equation and in the labor-consumption tradeoff, for instance when inflation changes the optimal decisions.

2. ***[Compulsory question]*** Recall the log-linearized formulas[12] for (a) the dynamic IS-equation (linking the present and future output gaps and interest rate), (b) the New-Keynesian Phillips Curve (linking present and future inflation rates and aggregate demand) and (c) the interest rate rule[13] (showing how monetary policy evolves). Describe these relations and their economic intuitions.
For the monetary policy rule, briefly explain the Taylor principle.

Solution:

1. In the New Keynesian models, the inter- and intra-temporal First Order Conditions rewrite:

$$Q_t = \beta\Big(\frac{P_t}{P_{t+1}}\Big)\Big(\frac{C_t}{C_{t+1}}\Big)^{\sigma} \qquad\qquad C_t^{\sigma}\, L_t^{\varphi} = \frac{W_t}{P_t}$$

Now, in the Euler equation, a wedge appears due to price level changes: when prices are expected to increase in t+1 (again, here there should be an expectation "$\mathbb{E}_t$" sign), then – everything else held constant – consumption C_t should increase: consumption smoothing is weighted by the change in the price level. Moreover, if nominal wages fall but prices take time to adjust, then the real wage falls and households decrease their consumption and/or their labor supply.

2. Noting that $i_t = -\log(Q_t)$ and $\rho = -\log\beta$, the Euler equation above yields, through its log-linearized version, the ***Dynamic IS equation***[14]:

$$\hat{c}_t = \mathbb{E}_t(\hat{c}_{t+1}) - \frac{1}{\sigma}(i_t - \mathbb{E}_t(\pi_{t+1}) - \rho) \qquad \text{Euler equation}$$
$$\widetilde{y}_t = \mathbb{E}_t(\widetilde{y}_{t+1}) - \frac{1}{\sigma}(i_t - \mathbb{E}_t(\pi_{t+1}) - r_t^n) \qquad \text{IS equation}$$
$$\text{with } r_t^n = \rho + \sigma\,\psi_{yz}^n\ \mathbb{E}_t(\Delta z_{t+1}) \qquad \text{Natural int. rate}$$

The second line uses goods market clearing, $C_t = Y_t$, and measures output as a gap, $\widetilde{y}_t = y_t - y_t^n$, relative to the natural level of output y_t^n that would prevail under flexible prices; ψ_{yz}^n is the elasticity of

[11] This exercise is advised for those who feel the need to practice.
[12] Simply state the three equations, no need to derive them! The main textbook for this type of model is: J. Gali - Monetary Policy, Inflation and the Business cycle
[13] There are different types of monetary policy rule, choose one and explain it.
[14] This time we give the stochastic version, with expectation signs.

that natural level to technology z_t.
Behind the IS equation lie the usual mechanisms of the standard Euler equation. However, the impact of monetary policy is clearer (it is a monetary model after all): when the real interest rate rises above its natural level, this has a negative impact on consumption (agents save instead of consuming), and the output gap widens. Again, because of the consumption smoothing motive, future consumption affects today's consumption linearly: agents' expectations about future policy therefore have a critical impact – which is what creates the "forward guidance" puzzle, where future policy decisions have oversized effects today.

The ***New-Keynesian Phillips Curve*** is derived from the definition of marginal cost (directly coming from the labor-consumption tradeoff) and from the price setting of firms:

$$
\begin{aligned}
w_t - p_t &= \sigma \hat{c}_t + \varphi \hat{\ell}_t && \text{labor-cons. tradeoff} \\
mc_t &= (w_t - p_t) - mpl_t && \text{effective marginal cost} \\
\widehat{mc}_t &= \big(\sigma + \tfrac{\varphi+\alpha}{1-\alpha}\big)(y_t - y_t^n) = \psi_{mc}\, \widetilde{y}_t && \text{"marginal cost" gap} \\
\pi_t &= \beta\, \mathbb{E}_t(\pi_{t+1}) + \lambda\, \widehat{mc}_t && \text{firms price setting} \\
\pi_t &= \beta\, \mathbb{E}_t(\pi_{t+1}) + \kappa\, \widetilde{y}_t && \text{New Keynesian Phillips Curve}
\end{aligned}
$$

with $\kappa = \lambda\, \psi_{mc}$. Behind the NKPC, the firms' pricing decisions have an impact on inflation: when aggregate demand increases, raising output above its flexible price level, it increases the disutility of labor (households dislike working more than at the steady state level) and, through workers-firms bargaining, wages increase and so do marginal costs. However, only a fraction $1 - \theta$ of firms can reset their price in a given period (Calvo pricing) – the coefficient λ of the NKPC is not that fraction, but a slope proportional to $\frac{(1-\theta)(1-\beta\theta)}{\theta}$, which decreases with the degree of price stickiness θ. Moreover, when firms expect future inflation (and a future rise in demand and marginal costs), they know that some of them will not be able to update their prices when the shock arrives, and prefer raising their prices now: this creates an expectation-driven "inflationary spiral".

The system is closed by a ***monetary policy rule***. A standard choice is the Taylor rule, which makes the nominal rate respond to inflation and to the output gap:

$$
i_t = \rho + \phi_\pi\, \pi_t + \phi_y\, \widetilde{y}_t + v_t \qquad\qquad \phi_\pi, \phi_y \geq 0
$$

where v_t is an exogenous monetary policy shock. The ***Taylor principle*** is the requirement that $\phi_\pi > 1$: the central bank must raise the *nominal* rate by more than one-for-one with inflation, so that the *real* rate $i_t - \mathbb{E}_t(\pi_{t+1})$ rises when inflation rises. Otherwise a rise in expected inflation lowers the real rate, which stimulates demand and raises inflation further: the economy has no nominal anchor, and the model admits self-fulfilling paths (indeterminacy) in which inflation moves without any change in fundamentals.

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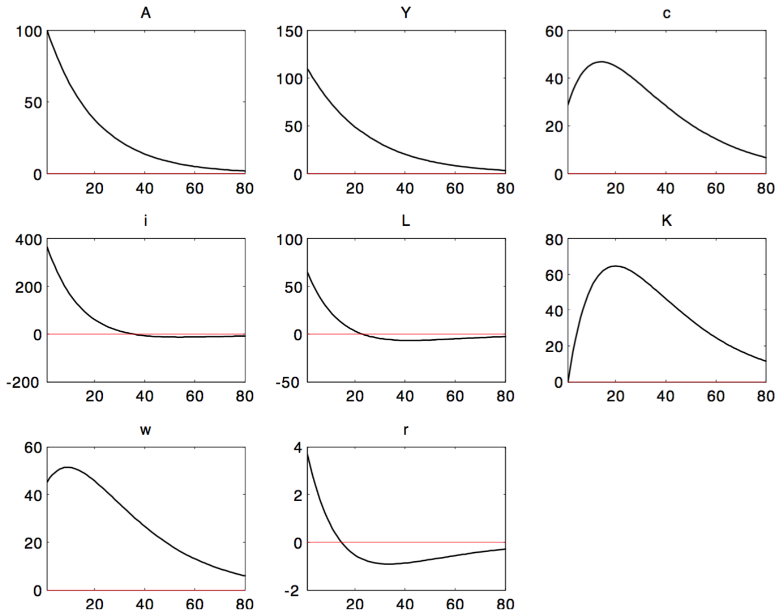
5 The dynamics of the two models (12 pts) Impulse response functions (IRF) to a technology shock

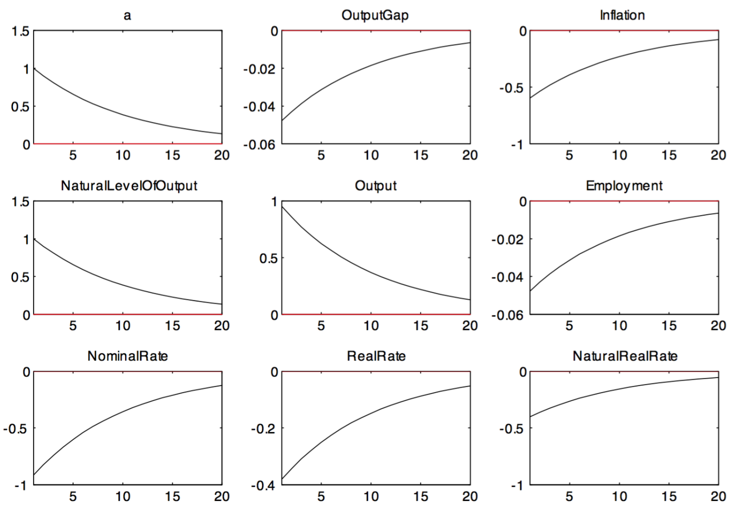
This is the response of the two standard models to a technology shock.[15]

Figure 1: *Baseline RBC model*



The relations structuring the first (RBC) model are the Euler equation, the labor-consumption tradeoff, the definition of output (Cobb-Douglas technology, and $Y_t = C_t + I_t$), the law of motion of capital ($K_{t+1} = (1-\delta)K_t + I_t$), the factor prices w_t and r_t set equal to the marginal productivities of the factors, and an AR(1) process for TFP.

Figure 2: *Baseline NK model*



The relations structuring the second (NK) model are the IS equation (or Euler equation), the

[15] *Notation in the figures.* The plots were produced in Dynare/Matlab, with notations differing slightly from the ones used previously. In the *RBC* figure: "A" is the TFP Z_t, "Y" is output Y_t, "c" is consumption C_t, "i" is investment I_t, "L" is labor L_t, "K" is capital K_t, "w" is the real wage w_t and "r" is the interest rate r_t. In the *NK* figure: "a" is the (log) TFP z_t, "NaturalLevelOfOutput" is y_t^n, "Output" is y_t, "OutputGap" is the output gap $\widetilde{y}_t = y_t - y_t^n$, "Inflation" is π_t, "Employment" is $\hat{\ell}_t$, "NominalRate" is i_t, "RealRate" is the ex-ante real rate $i_t - \mathbb{E}_t(\pi_{t+1})$ and "NaturalRealRate" is r_t^n. In both figures, all variables are plotted as deviations from their steady-state value, and the horizontal axis is the number of periods after the shock.

NK-Phillips Curve, the monetary policy rule for the nominal interest rate, the definitions of the natural level of output and of the natural rate of interest (the ones that would prevail in the flexible price economy), the output gap (effective output minus natural output), the real rate as the nominal rate net of expected inflation, and employment.

Describe the economic mechanisms underlying the two sets of Impulse Response Functions (IRF) following a technology shock, first in the Real-Business-Cycle model, and second in the New-Keynesian model.

Solution:

In the RBC model, when technology momentarily improves above trend, the marginal product of both capital and labor increases, which leads to higher wages and higher interest rates. Agents respond to the higher wage rate by increasing their labor supply. Output has to increase. Since agents want to have a smooth consumption pattern, aggregate consumption initially increases (proportionally) by less than output and, hence, investment increases (proportionally) by more than output. Capital slowly accumulates over time. The accumulation of capital lowers its marginal product and leads to an interest rate below its steady state level. Once technology is almost back to trend, there remains a wealth effect due to the higher capital stock. Agents therefore consume more while working slightly less than in steady state.

In the NK model, the favorable TFP shock raises the natural level of output – the one that would prevail under flexible prices – and lowers firms' marginal costs. It also lowers the natural rate of interest: since the shock is mean-reverting, technology is expected to fall back, $\mathbb{E}_t(\Delta z_{t+1}) < 0$, and $r_t^n = \rho + \sigma\psi_{yz}^n \, \mathbb{E}_t(\Delta z_{t+1})$ drops below ρ. However, prices are sticky and not all firms can adjust them (some would like to cut their price but cannot, because of Calvo-style staggered price setting). Demand therefore does not rise as much as productivity: output does increase, but by less than its natural level, so the output gap turns *negative*, and firms need fewer hours to produce it – employment falls. Notice that the improvement in technology is partly accommodated by monetary policy, which eases the nominal interest rate. This is not sufficient to close the negative output gap, and the fall in marginal costs yields deflation.

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6 Stochastic income processes (30 pts) Random walk, Markov chain, Martingale, AR(1), e tutti quanti

The first half of this problem set was written in *general equilibrium* and in a *deterministic* environment: households and firms interacted, and Y_t denoted aggregate output. In this second half, we zoom in on a single household in *partial equilibrium* – prices and the income process are taken as given – and we introduce *risk*. Accordingly, the notation shifts: from now on Y_t denotes the household's exogenous *income* ($\sim$ labor income $w_t L_t$, now random, and labor supply is no longer chosen by agents). This is the notation used in Sections 7 and 8, and in the consumption-saving and heterogeneous-agent literature more broadly.

We now introduce a simple shock for income $Y_t = Y(s_t)$. It fluctuates randomly according to a stochastic process, where s_t is a "state of the world" (or alea) at date t that belongs to the information set $\mathcal{I}_t$ (what is known to be possible to happen at date t)[16]. More specifically, a sequence of state-of-the-world $\{s_0, s_1, \dots, s_{t-1}, s_t\} = s^t$ defines a "history" s^t of shocks until t (this is the economist way of writing things compactly).

Putting some structure on this process $Y(s^t)$ (making it depend on the past), we consider an Autoregressive – AR(1) – process for income:

$$\begin{cases} Y_t(s^t) &= \rho_Y Y_{t-1}(s^{t-1}) + \varepsilon(s_t) \\ Y_{t_0} &= y \end{cases}$$

where $\rho_Y \in \mathbb{R}$ is a parameter and ε is an i.i.d. random variable (i.e. a function of the state of the world), that simply[17] takes three values: $+1$ with proba p, -1 with proba q and 0 with proba $1-p-q$. The following questions (b.) to (g.) are independent from each other (but most of them need the answer of (a.)).

(a.) Compute the conditional expectation $\mathbb{E}(Y_t(s^t)|\mathcal{I}_{t-1})$, as a function of the parameters and note that this expectation is a function of a variable depending on s^{t-1}.
- Using the same logic, write the value $\mathbb{P}(Y_t(s^t) = z|\mathcal{I}_{t-1})$ (Hint: you may need to use indicator functions $\mathbb{1}\{W = w\}$ for a random variables W and values w to be determined).

Solution:
The conditional expectation of $Y_t(s^t)$ given the information $\mathcal{I}_{t-1}$ is :

$$\begin{aligned} \mathbb{E}(Y_t(s^t)|\mathcal{I}_{t-1}) &= \mathbb{E}(\rho_Y Y_{t-1}(s^{t-1}) + \varepsilon(s_t)|\mathcal{I}_{t-1}) \\ &= \rho_Y Y_{t-1}(s^{t-1}) + \mathbb{E}(\varepsilon(s_t)|\mathcal{I}_{t-1}) \\ &= \rho_Y Y_{t-1}(s^{t-1}) + \mathbb{E}(\varepsilon(s_t)) \\ &= \rho_Y Y_{t-1}(s^{t-1}) + p - q \end{aligned}$$

We pass from the first to second line because $Y_{t-1}(s^{t-1})$ is known at time $t-1$ (we already have information about its value at $\mathcal{I}_{t-1}$, i.e. $Y_{t-1}(s^{t-1})$ is measurable w.r.t. to $\mathcal{I}_{t-1}$ in measure theory language).
We get the third line because $\varepsilon(s_t)$ is independent of past shocks: the information $\mathcal{I}_{t-1}$ does not help

[16] In measure theory language, this information set is the $\sigma-$algebra generated by the stochastic process until t (and $\{\mathcal{I}_t\}_t$ is a filtration: $\mathcal{I}_{t-1} \subset \mathcal{I}_t \subset \mathcal{I}_{t+1}$), i.e. it implies that $Y(s_t)$ is measurable with respect to this $\sigma-$algebra: it doesn't include values that are "outside" this set (if measure theory is too abstract for you, don't worry and just ignore this last sentence/footnote!)

[17] If you find that too easy, feel free to consider that as a random variable $\varepsilon(s_t) \sim \mathcal{N}(\mu_Y, \sigma_Y)$

predict $\varepsilon(s_t)$, so the conditional expectation is the unconditional one.
Last line is the definition of the law of $\varepsilon(s_t)$ (it is $\mathbb{E}(\varepsilon(s_t)) = \mu_Y$ if you assume $\varepsilon(s_t) \sim \mathcal{N}(\mu_Y, \sigma_Y)$).
We see that this conditional expectation $\mathbb{E}(Y_t(s^t)|\mathcal{I}_{t-1})$ is a *function* of period s^{t-1} information $\mathcal{I}_{t-1}$.

Similarly, the conditional probability can also be computed with the same logic. We note that because $\mathbb{P}(Z = z|\mathcal{F}) = \mathbb{E}[\mathbb{1}\{Z = z\}|\mathcal{F}]$, the approach follows :

$$\begin{aligned}
\mathbb{P}(Y_t(s^t) = z|\mathcal{I}_{t-1}) &= \mathbb{P}(\rho_Y Y_{t-1}(s^{t-1}) + \varepsilon(s_t) = z|\mathcal{I}_{t-1}) \\
&= \mathbb{P}\big(\varepsilon(s_t) = 1 \ \&\ \rho_Y Y_{t-1}(s^{t-1}) = z-1\big|\mathcal{I}_{t-1}\big) + \mathbb{P}\big(\varepsilon(s_t) = 0 \ \&\ \rho_Y Y_{t-1}(s^{t-1}) = z\big|\mathcal{I}_{t-1}\big) \\
&\qquad + \mathbb{P}\big(\varepsilon(s_t) = -1 \ \&\ \rho_Y Y_{t-1}(s^{t-1}) = z+1\big|\mathcal{I}_{t-1}\big) \\
&= \mathbb{P}(\varepsilon(s_t) = 1)\times\mathbb{P}\big(\rho_Y Y_{t-1}(s^{t-1}) = z-1\big|\mathcal{I}_{t-1}\big) + \mathbb{P}(\varepsilon(s_t) = 0)\times\mathbb{P}\big(\rho_Y Y_{t-1}(s^{t-1}) = z\big|\mathcal{I}_{t-1}\big) \\
&\qquad + \mathbb{P}(\varepsilon(s_t) = -1)\times\mathbb{P}\big(\rho_Y Y_{t-1}(s^{t-1}) = z+1\big|\mathcal{I}_{t-1}\big) \\
&= p\mathbb{1}\{Y_{t-1}(s^{t-1}) = \tfrac{z-1}{\rho_Y}\} + (1-p-q)\mathbb{1}\{Y_{t-1}(s^{t-1}) = \tfrac{z}{\rho_Y}\} + q\mathbb{1}\{Y_{t-1}(s^{t-1}) = \tfrac{z+1}{\rho_Y}\}
\end{aligned}$$

We get the third line – i.e. we could separate the probability sign because the events are disjoint[18] To get the fourth line, we use the fact that the events inside the probability sign are independent[19] (as ε is i.i.d. and hence independent of Y_{t-1}). The last line is obtained because again information on $Y_{t-1}(s^{t-1})$ is included in $\mathcal{I}_{t-1}$ and we don't need to take expectation (because we know its value at $t-1$). Again, we realize that this conditional probability is a function of $Y_{t-1}(s^{t-1})$.

$\cdots$ ——— $\cdots$

(b.) Show what are the conditions on the parameters (ρ_Y, p and q, etc.[20]), for Y_t to be a martingale (or a sub-martingale, or a super-martingale).

———

Solution:

Let us first verify that the sequence has finite first moment : $\mathbb{E}\big[|Y_t|\big] = \mathbb{E}\big[|Y_{t-1} + \varepsilon|\big] \leq \mathbb{E}\big[|Y_{t-1}|\big] + \overbrace{\mathbb{E}\big[|\varepsilon|\big]}^{\leq 1}$ (triangle inequality). Hence if $\mathbb{E}\big[|Y_{t-1}|\big] < \infty$ then $\mathbb{E}\big[|Y_t|\big] < \infty$.
We can thus say (by recursion) that if $\mathbb{E}\big[|Y_{t_0}|\big] = \mathbb{E}\big[|y|\big] = |y| < \infty$, then $\forall\, t \geq 1$, $\mathbb{E}\big[|Y_t|\big]$ has a finite first moment.

Now let us check the second condition. We computed the conditional expectation in question a):

$$\mathbb{E}(Y_t(s^t)|\mathcal{I}_{t-1}) = \rho_Y Y_{t-1}(s^{t-1}) + p - q$$

Y_t is a martingale when $\rho_Y = 1$ and $p = q$, so that $\mathbb{E}(Y_t|\mathcal{I}_{t-1}) = Y_{t-1}$. This is called a random walk (more on this in questions (d.) and (g.)). Still with $\rho_Y = 1$, the process is a submartingale (it grows on average) if $p \geq q$, and a supermartingale (it falls on average) if $p \leq q$. In these two cases the process drifts up or down forever, which does not prevent it from having finite moments at every date: $\mathbb{E}\big[|Y_t|\big] < \infty, \forall\, t$ isn't the same as $\mathbb{E}\big[\sup_{t\geq 1}|Y_t|\big] < \infty$.
Careful when $\rho_Y \neq 1$: the supermartingale condition $\mathbb{E}(Y_t|\mathcal{I}_{t-1}) \leq Y_{t-1}$ reads $(p-q) \leq (1-\rho_Y)Y_{t-1}$, which depends on the sign and the size of Y_{t-1} and hence cannot hold for every realization. A mean-reverting AR(1) is therefore neither a super- nor a sub-martingale globally.

$\cdots$ ——— $\cdots$

[18] The events A and B are disjoint when $\mathbb{P}(A \cap B) = 0$ and we used $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B)$.

[19] The events A and B are independents, then $\mathbb{P}(A \cap B) = \mathbb{P}(A)\times\mathbb{P}(B)$.

[20] This includes μ_Y if you consider a Normal law for ε

(c.) Does this stochastic process respect the Markov property? Can we say that it is a Markov chain (i.e. Markov process)? What is the set of states? (Hint: this needs not be finite). Write the transition matrix as function of the parameters, for $\rho_Y = 1$.

Solution:
We check the memoryless property, namely that

$$\mathbb{P}\big(Y_t \in A \mid \mathcal{I}_{t-1}\big) := \mathbb{P}\big(Y_t \in A \mid Y_{t-1}, Y_{t-2}, \dots\big) = \mathbb{P}\big(Y_t \in A \mid \sigma(Y_{t-1})\big)$$

We computed the conditional probability on the RHS in question (a.), and we saw that it is a function that only depends on $Y_{t-1}(s^{t-1})$, and doesn't require the knowledge (memory) of past events beside this value Y_{t-1}. Hence, $\{Y_t\}_t$ is a sequence of random variables respecting the Markov property, hence it is a Markov process, or discrete-time Markov chain.

Let us assume that $y \in \mathbb{Z}$ and $\rho_Y = 1$. Since ε takes the values $\{-1, 0, +1\}$, the state space is discrete but infinite, and equal to $\mathbb{Z}$: starting from y, the process can move up or down by one unit at each date, so it is *not* bounded below by 0 even if y is positive. If $y \notin \mathbb{Z}$, the state space is the shifted lattice $y + \mathbb{Z}$; if $\rho_Y \neq 1$, it is still countable but no longer a lattice. If we assume instead that $\varepsilon \sim \mathcal{N}(\mu_Y, \sigma_Y)$, the process is still Markovian (because ε is i.i.d. over time) but the state space becomes $\mathbb{R}$.

The transition matrix is $\{P_{i,j}\}_{i,j\in\mathbb{N}}$ with :

$$P_{i,j} = \mathbb{P}(Y_t = j \mid Y_{t-1} = i) = \begin{cases} q & \text{if } j = i-1 \\ (1-p-q) & \text{if } j = i \\ p & \text{if } j = i+1 \end{cases}$$

If $\varepsilon \sim \mathcal{N}(\mu_Y, \sigma_Y)$, then $\frac{Y_t - \rho_Y Y_{t-1} - \mu_Y}{\sigma_Y} \sim \mathcal{N}(0,1)$, and hence the Markov kernel would be

$$p(dx, y_{t-1}) = \phi\Big(\frac{x - \rho_Y y_{t-1} - \mu_Y}{\sigma_Y}\Big) dx$$

with $\phi(\cdot)$ the density of the standard normal distribution. This is a hint for question f.

$\cdots$ ——— $\cdots$

(d.) Assume that $\rho_Y = 1$. This process is called a random walk. Suppose that there are many agents with the same income process, starting from $y \in \mathbb{Z}$. What is the law of motion of the distribution π_t over the income "states"?
- Give explicitly the distribution of income at dates $t = 1$ and $t = 2$, when $y = 2$.
- Do you expect this process to have a stationary distribution over income states? (if yes, which one? if not, why not?)

Solution:
Let us denote $\pi_t = [\dots \pi_{i,t} \dots]$, an infinite row vector, and $P = \{P_{i,j}\}_{i,j\in\mathbb{Z}}$ the transition matrix. The law of motion is given by the equation:

$$\pi_t = \pi_{t-1} P$$

More explicitly, we have :

$$\pi_{i,t} = \pi_{i-1,t-1}\, p + \pi_{i,t-1}(1-p-q) + \pi_{i+1,t-1}\, q \qquad \forall i$$

When $y = 2$, the distribution is concentrated at 2: $\pi_0 = [0, \dots, 0, 1, 0, \dots]$.
At time $t = 1$, the distribution is $\pi_1 = [0 \dots 0, q, 1-p-q, p, 0, \dots]$, where the three values sit in columns

1, 2 and 3. Similarly $\pi_2 = [\ldots, \pi_{i,2}, \pi_{i+1,2}, \ldots]$ is:

$$\pi_{i,2} = \begin{cases} 0 & \text{if } i < 0 \\ q^2 & \text{if } i = 0 \\ q(1-p-q) + (1-p-q)q & \text{if } i = 1 \\ qp + (1-p-q)^2 + pq & \text{if } i = 2 \\ 2p(1-p-q) & \text{if } i = 3 \\ p^2 & \text{if } i = 4 \\ 0 & \text{if } i > 4 \end{cases}$$

where, for example, the mass at $i = 2$ is the sum of the mass of agents moving down and then up, of those moving up and then down, and of those staying in place for two periods.

The distribution keeps spreading out over time, and there is *no* stationary probability distribution: the chain on $\mathbb{Z}$ is null recurrent when $p = q$, and transient otherwise. Total mass is always 1, but it escapes to infinity, so $\pi_{i,t} \to 0$ for every fixed state i without the limit being a distribution. This is why, in general, heterogeneous-agent models need either a mean-reverting income process ($\rho_Y < 1$) or a bound on the state space to deliver a well-defined stationary distribution.

$\cdots$ ——— $\cdots$

(e.) Assume that $\rho_Y < 1$. Write the expression of $Y(s^t)$ as a function of past shocks and parameters.

- In the VAR language, rewrite this with lag operators. Moreover, if the first date was $t_0 \to -\infty$, can you reexpress this process as a MA(∞) process? As a consequence, can you rewrite this lag polynomial in a more compact way? (Hint: try to "inverting" an operator "as if" it was a standard division and use the rule for geometric series).
- In this setting, (or in the case where $t_0 = 0$ and $y = 0$) can you compute the variance $\mathbb{V}\text{ar}(Y_t)$ and the covariance $\mathbb{C}\text{ov}(Y_t, Y_{t-1})$? (Hint: recall that $\mathbb{V}\text{ar}(Z_t) = \mathbb{E}(Z_t^2) - \mathbb{E}(Z_t)^2$, for random variables Z_t that admit a finite second moment).
- Using the answer for the last subquestion, can you propose a method of moments (GMM) to estimate p, q and ρ_Y.

———

Solution:

(i) Using a recursion formula, we have:

$$Y_t = \rho_Y Y_{t-1} + \varepsilon_t = \rho_Y^2 Y_{t-2} + \rho_Y \varepsilon_{t-1} + \varepsilon_t = \cdots = \rho_Y^{t-t_0} Y_{t_0} + \sum_{k=0}^{t-t_0-1} \rho_Y^k \varepsilon_{t-k} \tag{*}$$

(ii) Using Lag operators, such that $Y_{t-1} = LY_t$, and $Y_{t-k} = L^k Y_t$ we have:

$$Y_t = \rho_Y L Y_t + \varepsilon_t = \rho_Y^2 L^2 Y_t + \rho_Y L \varepsilon_t + \varepsilon_t = \cdots = \rho_Y^{t-t_0} L^{t-t_0} Y_t + \sum_{k=0}^{t-t_0-1} \rho_Y^k L^k \varepsilon_t$$

If $t_0 \to -\infty$, and because $\rho_Y < 1$ (s.t. we have $\rho_Y^{t-t_0} \to 0$), the power series converges and the initial term Y_{t_0} fades away. Hence we have:

$$Y_t = \sum_{k=0}^{\infty} \rho_Y^k L^k \varepsilon_t$$

Moreover, heuristically, because we know that the sum of the power series is : $\sum_{k=0}^{T} \alpha^k e = \frac{(1-\alpha^{T+1})e}{1-\alpha}$, we can intuitively obtain:

$$\begin{aligned} Y_t &= \lim_{t_0 \to -\infty} \sum_{k=0}^{t-t_0-1} \rho_Y^k L^k \varepsilon_t \\ &= \lim_{t_0 \to -\infty} (1 - \rho_Y L)^{-1} (1 - \rho_Y^{t-t_0}) \varepsilon_t \\ &= (1 - \rho_Y L)^{-1} \varepsilon_t \end{aligned}$$

Another way to see it :

$$Y_t = \rho_Y Y_{t-1} + \varepsilon_t \qquad \Rightarrow \qquad Y_t - \rho_Y Y_{t-1} = (1-\rho_Y L)Y_t = \varepsilon_t$$
$$\Rightarrow \qquad Y_t = (1-\rho_Y L)^{-1}\varepsilon_t$$

(iii) To compute the variances, we use the equation $(*)$ and the fact that $\mathbb{V}\mathrm{ar}(Z) = \mathbb{E}(Z^2) - \mathbb{E}(Z)^2$. First, see that because either the first term fades away (when $t_0 = -\infty$) or is null when $y = 0$, we have the second term:

$$\begin{aligned}
\mathbb{E}[Y_t] &= \mathbb{E}\Big[\sum_{k=0}^{t-t_0-1} \rho_Y^k \varepsilon_{t-k}\Big] \\
&= \sum_{k=0}^{t-t_0-1} \rho_Y^k \mathbb{E}[\varepsilon_{t-k}] = \sum_{k=0}^{t-t_0-1} \rho_Y^k (p-q) \\
&= \frac{1-\rho_Y^{t-t_0}}{1-\rho_Y}(p-q)
\end{aligned}$$

Now, for the second moment, we know that $\mathbb{E}[\varepsilon_t^2] = p+q$ and $\mathbb{E}[\varepsilon_t] = p-q$. It is easier to work directly with the *centered* shocks: write $\varepsilon_t = \mu_Y + \eta_t$ with $\mu_Y = \mathbb{E}[\varepsilon_t] = p-q$ and $\mathbb{E}[\eta_t] = 0$, $\mathbb{V}\mathrm{ar}[\eta_t] = \mathbb{V}\mathrm{ar}[\varepsilon_t] = \sigma_Y^2 = (p+q) - (p-q)^2$. ***Careful***: independence gives $\mathbb{E}[\varepsilon_{t-k}\varepsilon_{t-k'}] = \mathbb{E}[\varepsilon_{t-k}]\mathbb{E}[\varepsilon_{t-k'}] = \mu_Y^2$ for $k \neq k'$, which is *not* zero unless $p = q$ – this is why we center the shocks first. Using the linearity of the variance operator on independent terms:

$$\begin{aligned}
\mathbb{V}\mathrm{ar}[Y_t] &= \mathbb{V}\mathrm{ar}\Big[\sum_{k=0}^{t-t_0-1} \rho_Y^k \varepsilon_{t-k}\Big] = \sum_{k=0}^{t-t_0-1} \rho_Y^{2k} \mathbb{V}\mathrm{ar}[\varepsilon_{t-k}] \\
&= \frac{1-\rho_Y^{2(t-t_0)}}{1-\rho_Y^2}\sigma_Y^2 \quad \xrightarrow[t_0\to-\infty]{} \quad \frac{\sigma_Y^2}{1-\rho_Y^2} = \frac{(p+q)-(p-q)^2}{1-\rho_Y^2}
\end{aligned}$$

Similarly, for the covariance, only the terms with the same shock survive (by changing the indices $k'' = k'+1$, the shock η_{t-k} appears in Y_{t-1} with one power of ρ_Y less):

$$\begin{aligned}
\mathbb{C}\mathrm{ov}[Y_t, Y_{t-1}] &= \mathbb{C}\mathrm{ov}\Big[\sum_{k=0}^{t-t_0-1} \rho_Y^k \eta_{t-k}\,,\ \sum_{k'=0}^{t-t_0-2} \rho_Y^{k'} \eta_{t-1-k'}\Big] \\
&= \sum_{k''=1}^{t-t_0-1} \rho_Y^{k''}\rho_Y^{k''-1}\sigma_Y^2 = \rho_Y \frac{1-\rho_Y^{2(t-t_0-1)}}{1-\rho_Y^2}\sigma_Y^2 \quad \xrightarrow[t_0\to-\infty]{} \quad \rho_Y\frac{\sigma_Y^2}{1-\rho_Y^2}
\end{aligned}$$

so that $\mathbb{C}\mathrm{ov}[Y_t, Y_{t-1}] = \rho_Y \mathbb{V}\mathrm{ar}[Y_t]$: the first-order autocorrelation of an AR(1) is simply ρ_Y. These expressions hold verbatim in the gaussian case $\varepsilon \sim \mathcal{N}(\mu_Y, \sigma_Y)$, with σ_Y^2 now the variance of the normal law.

(iv) We have the following moments conditions:

$$\mathbb{E}[Y_t] = \frac{\mu_Y}{1-\rho_Y} = \frac{p-q}{1-\rho_Y} \qquad \mathbb{V}\mathrm{ar}[Y_t] = \frac{\sigma_Y^2}{1-\rho_Y^2} = \frac{(p+q)-(p-q)^2}{1-\rho_Y^2}$$
$$\mathbb{C}\mathrm{ov}[Y_t, Y_{t-1}] = \rho_Y\frac{\sigma_Y^2}{1-\rho_Y^2} = \rho_Y\frac{(p+q)-(p-q)^2}{1-\rho_Y^2}$$

Let us call $\overline{Y}$, $var(Y_t)$ and $cov(Y_t, Y_{t-1})$ the empirical counterparts. An estimator given by the generalized method of moment should search for the parameters p, q and ρ_Y such that the theoretical moments match the empirical ones. Note that here the system can even be inverted by hand: $\rho_Y = cov(Y_t, Y_{t-1})/var(Y_t)$, then $\sigma_Y^2 = (1-\rho_Y^2)var(Y_t)$ and $\mu_Y = (1-\rho_Y)\overline{Y}$, and finally p and q from

$\mu_Y = p-q$ and $\sigma_Y^2 = (p+q)-(p-q)^2$. More generally, one minimizes the quadratic loss function:

$$\min_{p,q,\rho_Y} \alpha_1\Big[\frac{p-q}{1-\rho_Y} - \overline{Y}\Big]^2 + \alpha_2\Big[\frac{(p+q)-(p-q)^2}{1-\rho_Y^2} - var(Y_t)\Big]^2 + \alpha_3\Big[\rho_Y\frac{(p+q)-(p-q)^2}{1-\rho_Y^2} - cov(Y_t,Y_{t-1})\Big]^2$$

$$\min_{p,q,\rho_Y} (m(p,q,\rho_Y)-\mathcal{M}_N)'W(m(p,q,\rho_Y)-\mathcal{M}_N)$$

with $m(p,q,\rho_Y)$ is the vector of theoretical moments and $\mathcal{M}_N$ is the empirical counterpart for a sample of N observations. The second line is more general with a weighting matrix to balance the different terms in functions of their variances.

Here we have 3 moments and 3 parameters, so we are exactly identified. Adding more moments (like $\mathbb{C}\text{ov}(Y_t, Y_{t-k}), k \geq 2$), i.e. increasing the degree of overidentification, can lead to small sample bias (there is a tradeoff sample-size/number of moments conditions), and you will see more on that in your econometrics courses.

...——— ...

(f.) Assume $\varepsilon \sim \mathcal{N}(\mu_Y, \sigma_Y)$ for this question. Suppose you had data on $Y_0, Y_1, \ldots, Y_T$, and you would like to estimate the parameters ρ_Y, μ_Y and σ_Y using Maximum Likelihood estimation (MLE). Write the (log-)likelihood, i.e. the objective of this maximization problem.

———

Solution:
The vector of parameters is $\theta = (\rho_Y, \mu_Y, \sigma_Y)$. Conditioning on the first observation y_0, the log-likelihood is:

$$\begin{aligned}
\ell(\theta) &= \log \prod_{t=1}^{T} \phi\Big(\frac{y_t - \rho_Y y_{t-1} - \mu_Y}{\sigma_Y}\Big) \\
&= \log \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\Big(-\frac{(y_t - \rho_Y y_{t-1} - \mu_Y)^2}{2\sigma_Y^2}\Big) \\
&= -\frac{T}{2}\log(2\pi) - T\log(\sigma_Y) - \sum_{t=1}^{T} \frac{(y_t - \rho_Y y_{t-1} - \mu_Y)^2}{2\sigma_Y^2} \\
&= -\frac{T}{2}\log(2\pi) - T\log(\sigma_Y) - \frac{T}{2\sigma_Y^2}\mu_Y^2 - \frac{\mu_Y\rho_Y}{\sigma_Y^2}\sum_{t=1}^{T} y_{t-1} + \frac{\mu_Y}{\sigma_Y^2}\sum_{t=1}^{T} y_t - \frac{\rho_Y^2}{2\sigma_Y^2}\sum_{t=1}^{T} y_{t-1}^2 + \frac{\rho_Y}{\sigma_Y^2}\sum_{t=1}^{T} y_{t-1}y_t - \frac{1}{2\sigma_Y^2}\sum_{t=1}^{T} y_t^2
\end{aligned}$$

One could take the first order conditions w.r.t. μ_Y, σ_Y and ρ_Y, to obtain three equations that allow to determine the MLE for these parameters. However, for more complicated problems it is frequent to optimize this log-likelihood numerically!

...——— ...

(g.) Suppose ε is again the original three states $\{+1, 0, -1\}$ random variable, with probabilities p, q such that $\mathbb{E}(\varepsilon) = 0$ and $\mathbb{V}\text{ar}(\varepsilon) = 1$ (what are these values of p and q?). Assume that $Y_0 = 0$ and that the process Y_t is both a martingale and a Markov chain (i.e. we have the properties of questions (b.) and (c.)).
- What do you expect this process to look like when the time period becomes infinitesimally small, i.e. $\Delta t = t_{i+1} - t_i \to 0$. Can you give the name of this stochastic process?
- More formally, define the rescaled process

$$W^{(n)}(t) = \frac{Y_{\lfloor nt \rfloor}}{\sqrt{n}}, \qquad t \in [0,1]$$

Can you determine if $W^{(n)}(1)$ converge in distribution toward a specific random variable? If yes, which one? What is the name of this theorem?
- (Advanced) Can one show this convergence for every t, and hence determine the law of $W(t)$ at the limit for n? What is the name of this theorem?

Solution:

The conditions $\mathbb{E}(\varepsilon) = p - q = 0$ and $\mathbb{V}\mathrm{ar}(\varepsilon) = \mathbb{E}(\varepsilon^2) = p + q = 1$ give $p = q = \frac{1}{2}$: the middle state has probability zero, and ε degenerates to a symmetric ± 1 variable, i.e. Y_t is a simple random walk.

When the time steps converge to zero, we obtain a Brownian motion (or Wiener process), the simplest stochastic process that is at the same time a Markov process, a martingale and continuous. The more general Lévy processes (jump and diffusion processes) are also Markov processes and can be martingales, but need not be continuous.

Formally, since $Y_0 = 0$ and $\rho_Y = 1$, $W^{(n)}(1) = \frac{Y_n}{\sqrt{n}} = \frac{\sum_{k=1}^{n} \varepsilon_k}{\sqrt{n}}$, where the ε_k are i.i.d. with mean zero and unit variance. Hence, by the central limit theorem, $W^{(n)}(1)$ converges in distribution towards a standard gaussian random variable $\mathcal{N}(0, 1)$.

Now, for every $t \in [0, 1]$, $W^{(n)}(t)$ converges to the Brownian motion W_t, whose law is $\mathcal{N}(0, t)$: its standard deviation grows with the square root of time, $\sqrt{t}$. This convergence in distribution happens in the (infinite dimensional) Skorokhod space $\mathcal{D}([0, 1])$ (say, the space of continuous processes on $[0, 1]$). This is the Donsker theorem. It is also what delivers the limiting distribution of the Kolmogorov-Smirnov statistic, used to test the cdf $F(x)$ of a random variable X, which is itself a function in $\mathcal{D}(\mathbb{X})$.

The Brownian motion being the continuous-time limit of any random walk explains its ubiquity in continuous-time models in economics and finance.

…——— …

7 The Household problem in Arrow-Debreu economies (10 pts) State-dependent consumption & portfolio choice

Using a stochastic process for income, we assume that the market is complete in the Arrow-Debreu (AD) sense: for each history s^t and each state of the world s_{t+1}, there exists an asset $A_{t+1}(s_{t+1}|s^t)$ that yields one unit of consumption in state s_{t+1}, and that can be purchased at date t after history s^t, at the price $Q_t(s_{t+1}|s^t)$ (for compact notation: $A_{t+1}(s_{t+1}|s^t) \equiv A_{t+1}(s^{t+1})$ and $Q_t(s_{t+1}|s^t) \equiv Q_t(s^{t+1})$). Now, given this complete market structure, the household maximizes its expected utility, given the probability $\pi(s^t)$ of history s^t.

$$\max_{\{C_t(s^t)\}_0^\infty,\{A_{t+1}(s^{t+1})\}_0^\infty} \sum_{t=0}^{\infty} \sum_{s^t \subset \mathcal{I}_t} \beta^t \pi(s^t) U\big(C_t(s^t)\big)$$

$$\text{s.t.} \qquad C_t(s^t) + \sum_{s_{t+1}} Q_t(s_{t+1}|s^t) A_{t+1}(s_{t+1}|s^t) \leq A_t(s^t) + Y_t(s^t) \qquad \forall\, t \geq 0,\ \forall\, s^t$$

with $A_0(s^0)$ given, and a no-Ponzi condition holding state by state, as in Section 0.3. Using the Lagrangian method, derive the Euler equation and specify the stochastic discount factor.

Solution:

Note that this time the Lagrange multiplier is state-dependent, $\lambda_t(s^t)$. The Lagrangian is:

$$\mathcal{L}\big(C_{t(s^t)}, \{A_{t+1(s_{t+1}|s^t)}\}_{s_{t+1}\subset\mathcal{I}_{t+1}}, \lambda_{t(s^t)}\big) = \sum_{t=0}^{\infty} \sum_{s^t \subset \mathcal{I}_t} \Big\{\beta^t \pi(s^t) U\big(C_t(s^t)\big) - \lambda_t(s^t)\Big[C_t(s^t) + \sum_{s_{t+1}} Q_t(s_{t+1}|s^t) A_{t+1}(s_{t+1}|s^t) - A_t(s^t) - Y_t(s^t)\Big]\Big\}$$

1. Stationarity (w.r.t. $C_t(s^t)$ and $A_{t+1}(s_{t+1}|s^t)$):

$$\begin{aligned}
&[C_t] && \beta^t \pi(s^t)\, U'(C_t^*(s^t)) - \lambda_t(s^t) = 0 \qquad \forall\, t \geq 0,\ \forall\, s^t \\
&[A_{t+1}(s_{t+1}|s^t)] && -\lambda_t(s^t)\, Q_t(s_{t+1}|s^t) + \lambda_{t+1}(s^{t+1}) = 0 \\
&\text{(combining 1 \& 2)} && Q_t(s_{t+1}|s^t)\, \pi(s^t)\, U'(C_t^*(s^t)) = \beta\, \pi(s^{t+1})\, U'(C_{t+1}^*(s^{t+1}))
\end{aligned}$$

   Note that the asset $A_{t+1}(s_{t+1}|s^t)$ appears twice: it is bought at t after history s^t, at the price $Q_t(s_{t+1}|s^t)$, and it pays one unit of consumption at $t+1$ in the single history $s^{t+1} = (s^t, s_{t+1})$ – which is why only $\lambda_{t+1}(s^{t+1})$ enters the second line.
   The other optimality conditions (the ones that economists usually forget) are the same as before, but this time for every state of the world:

2. Primal feasibility: $F_t(s^t) \leq 0$ for $t \geq 0, \forall\ s^t \subset \mathcal{I}_t$
3. Dual feasibility: $\lambda_t(s^t) \geq 0 \quad \forall\, t \geq 0, \forall\ s^t \subset \mathcal{I}_t$
4. Complementarity: $\sum_{t=0}^{\infty} \sum_{s^t \subset \mathcal{I}_t} \lambda_t(s^t)\, F_t(s^t) = 0$ which is equivalent to $\lambda_t(s^t)\, F_t(s^t) = 0 \quad \forall\, t \geq 0, \forall\, s^t$

where $F_t(s^t) = C_t(s^t) + \sum_{s_{t+1}} Q_t(s_{t+1}|s^t) A_{t+1}(s_{t+1}|s^t) - A_t(s^t) - Y_t(s^t)$ is the date-t, history-s^t budget constraint.

We see that the Lagrange multiplier is proportional to the marginal utility of consumption *and* to the probability of the history s^t. Moreover, $\pi(s^{t+1}) = \pi(s^t, s_{t+1})$, so that $\frac{\pi(s^{t+1})}{\pi(s^t)} = \pi(s_{t+1}|s^t)$. Rewriting the previous equations more compactly:

$$Q_t(s_{t+1}|s^t) = \beta \frac{\pi(s^{t+1})\, U'(C_{t+1}(s^{t+1}))}{\pi(s^t)\, U'(C_t(s^t))} = \pi(s_{t+1}|s^t)\, \beta \frac{U'(C_{t+1})}{U'(C_t)} \qquad [\textbf{\textit{Euler equation}}]$$

In asset pricing (and dynamic/stochastic macro) the stochastic discount factor (s.d.f.) $m(s_{t+1}|s^t)$ is defined such that:

$$Q_t = \mathbb{E}[m(\widetilde{s}_{t+1})R_t(\widetilde{s}_{t+1})|\mathcal{I}_t]$$

where Q_t is the asset price and $R_t(\widetilde{s}_{t+1})$ its payoff – here $R_t(\widetilde{s}_{t+1}) = \mathbb{1}\{\widetilde{s}_{t+1} = s_{t+1}\}$, one unit in the state s_{t+1} and zero otherwise – with $\widetilde{s}_{t+1}$ the (dummy) variable over which we integrate in the expectation. Hence the s.d.f. is again the ratio of marginal utilities:

$$m(\widetilde{s}_{t+1}|s^t) = \beta\frac{U'(C_{t+1}(\widetilde{s}_{t+1}))}{U'(C_t(s^t))}$$

which makes sense, because in our (rather unrealistic) Arrow-Debreu economy there is one asset, and one asset price, for each state of the world s_{t+1}, and hence:

$$Q_t(s_{t+1}|s^t) = \mathbb{E}[m(\widetilde{s}_{t+1})\mathbb{1}\{\widetilde{s}_{t+1} = s_{t+1}\}|\mathcal{I}_t] = \pi(s_{t+1}|s^t)\, m(s_{t+1})$$

In the next exercise, we will remove this strong assumption of complete markets. ***Keep this formula in mind***: with complete markets there is a full menu of traded payoffs, one indicator $\mathbb{1}\{\widetilde{s}_{t+1} = s_{t+1}\}$ per state, i.e. *one asset per state*. The household therefore equates $\beta U'(C_{t+1})/U'(C_t)$ to a price *state by state*: the marginal rate of substitution is pinned down separately in every s_{t+1}, and the household can choose a consumption profile that no longer depends on which idiosyncratic shock it draws. This is *perfect risk sharing*, and it is why complete markets are the natural benchmark.

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8 The Household problem in incomplete markets (20 pts) Self-insurance and Dynamic programming

Consider the same problem as in the previous exercise. This time the market is incomplete: there is no asset contingent on each state of the world s_{t+1}. The only asset is a “risk-free bond”

$$\max_{\{C_t(s^t)\}_0^\infty,\{A_{t+1}(s^t)\}_0^\infty} \sum_{t=0}^{\infty} \sum_{s^t \subset \mathcal{I}_t} \pi(s^t)\beta^t U\big(C_t(s^t)\big)$$

$$\text{s.t.} \qquad C_t(s^t) + Q_t(s^t)A_{t+1}(s^t) \leq A_t(s^t) + Y_t(s^t) \qquad \forall\, t \geq 0,\ \forall\, s^t$$

with $A_0(s^0)$ given and a no-Ponzi condition, as in Section 0.3. To alleviate notations, we drop the history s^t where no confusion arises: all the variables chosen at date t are measurable w.r.t. $\mathcal{I}_t$, i.e. they may depend on the whole history s^t but not on the shocks s_{t+1} still to come.

For that, we will use a "Dynamic programming" procedure. Again, the household maximizes over its consumption and saving paths, and we will obtain an Euler equation.

The idea here is to solve[21] the "Bellman" equation: the objective can be rewritten as an iterative "Value function":

$$\begin{aligned}
V_t(A_t, Y_t) &= \max_{\{C_j, A_{j+1}\}_{j\geq t}} \mathbb{E}\Big[\sum_{j=t}^{\infty} \beta^{j-t} U(C_j) \big| \mathcal{I}_t\Big] \\
&= \max_{\{C_j, A_{j+1}\}_{j\geq t}} \sum_{j=t}^{\infty} \sum_{s^j \subset \mathcal{I}_j} \pi(s^j|s^t)\beta^{j-t} U(C_j) \\
\Leftrightarrow \qquad V_t(A_t, Y_t) &= \max_{C_t, A_{t+1}} \Big[U(C_t) + \beta\, \mathbb{E}\big(V_{t+1}(A_{t+1}, Y_{t+1}) \big| \mathcal{I}_t\big)\Big] \\
&= \max_{C_t, A_{t+1}} \Big[U(C_t) + \beta \sum_{s_{t+1} \subset \mathcal{I}_{t+1}} \pi(s_{t+1}|s^t)\, V_{t+1}\big(A_{t+1}, Y_{t+1}(s_{t+1})\big)\Big]
\end{aligned}$$

To solve this problem, you need to derive
(i) the KKT first-order conditions for this new (iterative) maximization problem[22], and
(ii) a new "Envelope condition". This second condition gives the derivative of the value function (w.r.t. A_t) and, through the chain rule, is linked to the "Envelope Theorem". In this theorem, when a function is maximized over a variable (V_t is maximized over A_{t+1} here) and hence optimal in this sense, its derivative w.r.t. that same variable is null.

Combining the different equations and manipulating the terms, show the Euler equation and show the similarities and differences with the Arrow-Debreu/complete market economy.

Solution:
(0.) The **Envelope Theorem** is the same tool as in Section 3: for a value function $v(x) = \max_{y\in K} f(x, y)$ whose optimum is interior and unique, $\nabla v(x) = \nabla_x f(x, y^*(x))$ – the derivative of v is the derivative of f with respect to its *first* (state) variable only, because $\nabla_y f(x, y^*(x)) = 0$ by the first-order conditions on the controls. We refer to the solution of Section 3 for the statement and the assumptions, and go directly to the economic problem here.

[21] During the course, we will discuss the general method to solve such problems (and prove that the sequential problem and its recursive formulation are equivalent).
[22] Note that we are back to the first "static" version of the KKT theorem (Section 0.1).

Let us come back to our **Bellman equation**:

$$V_t(A_t, Y_t) = \max_{C_t, A_{t+1}} \; U(C_t) + \beta\,\mathbb{E}\Big[V_{t+1}\big(A_{t+1}, Y_{t+1}(s_{t+1})\big)\Big|\mathcal{I}_t\Big]$$

Here, the value function depends on the state A_t, i.e. saving (in most macro models, it will be wealth in one form or another), and on the stochastic Y_t, i.e. income (most of the time an exogenous economic shock), while it is optimal w.r.t. the controls C_t, A_{t+1}.

(1.) First, we derive the **first-order conditions** of the problem at date t, which give the policy functions $C_t = \mathcal{C}(A_t, Y_t)$ and $A_{t+1} = \mathcal{A}(A_t, Y_t)$, both determined using the ... KKT theorem. Everything is state dependent – all the variables should carry s^t – but we only keep it on Y_t and $\lambda_t(s^t)$, to make clear where the shock comes from and how it affects the rest of the system:

$$\begin{aligned} F_t(C_t, A_{t+1}|s^t) &= C_t + Q_t A_{t+1} - A_t - Y_t(s^t) \leq 0 \\ \mathcal{L}_t(C_t, A_{t+1}, \lambda_t|s^t) &= U(C_t) + \beta\,\mathbb{E}\Big[V_{t+1}(A_{t+1}, Y_{t+1})\Big|\mathcal{I}_t\Big] - \lambda_t(s^t)\,F_t(C_t, A_{t+1}|s^t) \end{aligned}$$

Deriving w.r.t. C_t and A_{t+1} yields the following conditions (the primal and dual feasibility and the complementarity conditions are exactly the same as in the previous section, and we do not rewrite them):

$$\begin{aligned} &[C_t] && U'(C_t^*) - \lambda_t(s^t) = 0 \\ &[A_{t+1}] && \beta \sum_{s_{t+1}} \pi(s_{t+1}|s^t)\,\partial_A V_{t+1}\big(A_{t+1}^*, Y_{t+1}(s_{t+1})\big) - \lambda_t(s^t)\,Q_t = 0 \\ &\text{(combining 1 \& 2)} && Q_t\,U'(C_t^*) = \beta \sum_{s_{t+1}} \pi(s_{t+1}|s^t)\,\partial_A V_{t+1}\big(A_{t+1}^*, Y_{t+1}(s_{t+1})\big) \end{aligned}$$

(2.) Now, to derive the **envelope condition**, we need the derivative of V_t w.r.t. A_t (why? in the last equation we still have the unknown term $\partial_A V_{t+1}(A_{t+1}, Y_{t+1})$). Coming back to the Bellman equation, let us obtain $\partial_A V(A_t, Y_t)$. Here we can use the Envelope theorem, since V_t is already maximized over the two controls. ***But*** there is a trick: when the budget constraint binds, one of the controls can be expressed as a function of the state and of the other control,

$$C_t = \phi(A_t, A_{t+1}, Y_t) = -Q_t A_{t+1} + A_t + Y_t$$

so that the value function becomes an unconstrained maximum over A_{t+1}. The chain rule then gives:

$$\partial_A V(A_t, Y_t) = U'(C_t)\,\frac{\partial \phi(A_t, A_{t+1}, Y_t)}{\partial A_t} + \underbrace{\left[U'(C_t)\frac{\partial \phi(A_t, A_{t+1}, Y_t)}{\partial A_{t+1}} + \beta\,\mathbb{E}_t \frac{\partial V_{t+1}(A_{t+1}, Y_{t+1})}{\partial A_{t+1}}\right]}_{=0 \;\; (\text{FOC on } A_{t+1})} \frac{dA_{t+1}}{dA_t}$$

The bracket is exactly the first-order condition with respect to the control A_{t+1} – recall $\partial\phi/\partial A_{t+1} = -Q_t$ – hence it is zero at the optimum. Note that it is the *whole bracket* that vanishes, not the term $\beta\mathbb{E}_t[\partial_A V_{t+1}]$ alone, which equals $Q_t U'(C_t) \neq 0$. Therefore:

$$\begin{aligned} \partial_A V\big(A_t, Y_t(s_t)\big) &= U'(C_t)\,\frac{\partial \phi(A_t, A_{t+1}, Y_t)}{\partial A_t} \\ &= U'(C_t) \end{aligned}$$

In this very simple case, the marginal value of wealth (i.e. saving) equals the marginal utility of present consumption. This intuition holds in many macro models.

(3.) Since this condition holds at all dates and for all s^t, we can iterate it forward ($t \to t+1$) and obtain:

$$\partial_A V_{t+1}\big(A_{t+1}, Y_{t+1}(s_{t+1})\big) = U'(C_{t+1})$$

Replacing in the last first-order condition above:

$$Q_t\, U'(C_t) = \beta\, \mathbb{E}\big[\partial_A V_{t+1}\big(A_{t+1}, Y_{t+1}(s_{t+1})\big)\big|\mathcal{I}_t\big] = \beta\, \mathbb{E}\big[U'\big(C_{t+1}(s_{t+1})\big)\big|\mathcal{I}_t\big]$$

We recover the ***Euler equation*** (the intertemporal condition). It implies the asset pricing equation:

$$Q_t = \beta\, \mathbb{E}\Big[\frac{U'(C_{t+1})}{U'(C_t)}\Big|\mathcal{I}_t\Big]$$

so the stochastic discount factor is again the ratio of marginal utilities:

$$Q_t = \mathbb{E}[m(\widetilde{s}_{t+1})R_t(\widetilde{s}_{t+1})|\mathcal{I}_t] \qquad\qquad m(\widetilde{s}_{t+1}|s^t) = \beta\frac{U'(C_{t+1}(\widetilde{s}_{t+1}))}{U'(C_t(s^t))}$$

with again Q_t the asset price, R_t the payoff – equal to one in every state here, since the bond is risk-free – and $\widetilde{s}_{t+1}$ the (dummy) variable of integration. The s.d.f. is *stochastic*, as it depends on s^{t+1}: this is how households "anticipate" future shocks, the foundation of rational expectations.

Complete vs. incomplete markets – the point of Sections 7 and 8.

Note first what is *identical*: the kernel $m = \beta\, U'(C_{t+1})/U'(C_t)$ is the same function of marginal utilities in both economies. What changes is the menu of traded payoffs – and hence the consumption plan that ends up inside m. In the Arrow-Debreu economy, with complete markets, there is one Euler equation *per state of the world*: the household can trade a claim contingent on each s_{t+1}, equalize its marginal utility across states, and thereby insure its idiosyncratic income risk completely, so that consumption ends up depending only on the aggregate state, not on the household's own draw of Y_t (*perfect risk sharing*). With a single riskless bond, only *one* Euler equation holds, and it holds only *in expectation*: $Q_t U'(C_t) = \beta\mathbb{E}_t[U'(C_{t+1})]$. The household can no longer move resources across states, only across time, so consumption inherits the fluctuations of income and can only be smoothed by saving and dissaving – this is *self-insurance*, and it is why the buffer of assets A_t becomes a state variable of the problem. This contrast – perfect insurance vs. self-insurance – is the starting point of the Bewley-Huggett-Aiyagari literature on heterogeneous agents.

With the dynamic programming method, that seems a lot of work for one simple (and almost already known) equation ... but the good news is that this method is general and can be applied to all kinds of problems (especially when associated with the KKT theorem for inequality constraints). In the macro sequences, you will use the Dynamic programming approach to study macro models with incomplete markets, credit constraints, nominal rigidities, search & matching frictions, heterogeneous agents, etc. There will be two main methods: solving the Bellman equation numerically (approximating the value function in order to obtain the policy function), or making clever assumptions in order to derive the model analytically.

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