Problem Set 0 – Advanced Macroeconomics

Prerequisites: dynamic optimization, business cycles models and consumption-saving problems

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Due Date: Monday, 14th of September 2026

Description of this homework:

This "Homework 0" should make sure that all of you start the course with the same set of tools and background. It is aimed at recalling the central building blocks of the standard DSGE models[1] of modern macro: the Neoclassical Growth model (Ramsey–Cass–Koopmans and its stochastic extension by Brock–Mirman), the Real Business Cycle (RBC) model, the New-Keynesian (NK) model, and the consumption-saving problem with idiosyncratic risk (Bewley–Huggett–Aiyagari), which is the workhorse of the heterogeneous-agent literature.

This problem set is organized in three blocks. ***Part 0*** recalls the mathematical toolbox – constrained optimization and the Karush–Kuhn–Tucker (KKT) theorem – used throughout the course. ***Parts 1 to 5*** cover representative-agent business-cycle models in deterministic environments: the firm problem, the household problem solved first with both Lagrangian methods and dynamic programming, the New-Keynesian model, and the interpretation of IRFs in the two models. ***Parts 6 to 8*** introduce risk and market structure: stochastic processes for income, the household problem under complete markets with Arrow–Debreu securities, and incomplete-market "self-insurance" problem, both solved recursively. This last block requires probability theory, stochastic processes and dynamic optimization (i.e. control theory), which we will use extensively in the course.

This homework is *quite long*, so Section 6 is *optional*, as it relates more to macroeconometrics. In class, you will still be expected to be confortable with definitions/properties of stochastic processes, Markov Chains, AR(1) process, etc.

This work will not be graded. No need to work hundreds of hours on it: the main purpose is that all of you start with the same set of common knowledge, and that you can identify by yourself the tools you may want to review. For that reason, more explanations than usual are included in the questions themselves (to get you used to the notations and language). The number of points indicated at the beginning of each part is only a rough indication of the time and effort that each of them requires. For additional resources, see any standard textbook, e.g. D. Romer, *Advanced Macroeconomics*, J. Galí, *Monetary Policy, Inflation and the Business Cycle* (for Parts 4–5), or L. Ljungqvist & T. Sargent, *Recursive Macroeconomic Theory* (for Parts 6–8), or my Mathcamp lecture notes Part 1 and Part 2.
I will hold office-hours (3rd floor, EIEF main building) on Mon 07/09 and Wed 09/09, from 5 to 6 PM.

Rules for submission:

This problem set should be handed in on Monday, 14th of September 2026 before the morning class. Send an email to *thomas.bourany@eief.it* with the email subject "Macro Problem Set 0 [your name]". The solutions will be posted the day after, so no delay can be accepted. Problem sets should be handed in individually. However, it is strongly recommended to work in groups, especially for those of you who did not take the corresponding courses in the past years. Mention who you worked with at the top of your homework. Clarity and conciseness will be appreciated: for most questions, a few lines of derivation and 2-3 sentences of economic intuitions are enough.

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[1] DSGE stands for *Dynamic Stochastic General Equilibrium*: economic relations are dynamic (agents are forward-looking) and stochastic (the economy is subject to probabilistic "shocks" and agents form rational expectations). Moreover, the equilibrium is "general", as markets clear between aggregate demand – from households – and supply – from firms. [For simplicity, Parts 0-4 are derived in deterministic settings, and risk is introduced only from Part 6 onwards.]

0 Prerequisites (20 pts) Constrained optimization: Karush-Kuhn-Tucker Theorem

0.1 The static case

For an optimization problem, maximizing[2] a function $J(x)$ over a variable[3] x and subject to a set of M constraints $F_1(x) \leq 0 \dots F_M(x) \leq 0$. Recall the (Karush-)Kuhn-Tucker (KKT) theorem that states necessary and sufficient conditions for optimality.

0.2 The dynamic case in finite horizon

We extend the problem in a dynamic environment, often met in economic models in discrete time. Now the objective function is time-separable and the budget constraints and the variables are time-specific. Therefore, there are $T+1$ constraints[4].

$$\max_{\{x_t\}_{t=1}^{T+1}} \sum_{t=0}^{T} J_t(x_t)$$

$$\text{s.t.} \qquad F_t(x_t, x_{t+1}) \leq 0 \qquad \forall\, 0 \leq t \leq T \qquad (x_0 \text{ given})$$

Adapt the previous KKT conditions to this new environment.

0.3 The dynamic case in infinite horizon

We now let the horizon go to infinity. The objective becomes an infinite sum with infinitely many constraints, and time-specific Lagrange multipliers λ_t. We introduce a discount factor $\beta \in (0,1)$: discounting is what makes the objective finite, hence the problem well posed, when J is bounded.[5]

$$\max_{\{x_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t J(x_t)$$

$$\text{s.t.} \qquad F_t(x_t, x_{t+1}) \leq 0 \qquad \forall\, t \geq 0 \qquad (x_0 \text{ given})$$

It would be tempting to take the conditions of Section 0.2 and simply replace "T" by "∞", but an infinite-horizon problem is *not* simply a finite-horizon problem with large T.

(a.) To the finite-horizon problem, add the restriction $x_{T+1} \geq 0$ with multiplier ν. Write the stationarity/complementarity slackness condition w.r.t. x_{T+1}. What terminal condition does it imply? (There you can assume $\partial F_t(x_t, x_{t+1})/\partial x_{t+1} = 1$ for simplicity for the rest of Section 0). Interpret it. What is the natural candidate for the analogue of this condition, when $T \to \infty$? what is it called? Explain intuitions.

[2]Note, this is a maximization problem, maximizing "$J(x)$" is the "same" as minimizing "$-J(x)$". Formally: $\max_x J(x) = -\min_x -J(x)$

[3]Here, for simplicity, we suppose that $x \in \mathbb{R}$ or even $x \in \mathbb{R}^n$. The KKT theorem can be generalized to the case where $x \in X$ where X is a reflexive Banach space (therefore, x can be many different objects, for example a function! (in the case where X is an infinite-dimensional space)).

[4]Therefore, the Lagrange multipliers are time-specific: λ_t

[5]On a balanced growth path with growth rate g and CRRA preferences with parameter σ, the objective is unbounded and the problem is well posed if $\beta(1+g)^{1-\sigma} < 1$.

(b.) Are the KKT conditions still necessary for optimality of the path $\{x_t\}_{t=0}^{\infty}$ when $T \to \infty$? Are they still sufficient for optimality of $\{x_t\}_{t=0}^{\infty}$?
(c.) Explain the nuances and differences between that transversality condition and the No Ponzi condition, which arise in the consumption-saving problem of Section 2 below, where $x_t \equiv A_t$ is the asset holding, $1+r$ the gross return and $\lambda_t = \beta^t U'(C_t)$.

1 The Firm problem (10 pts) Profit maximization, factor prices, and constant returns to scale

1) Suppose the representative firm maximizes its profits, in a fully-competitive environment, adjusting its input factors (labor L_t and capital K_t). Output is $Y_t = Z_t F(K_t, L_t)$, where Z_t is the Total Factor Productivity (TFP), and profits are output net of the rental cost of capital ($R_t K_t$) and of wages ($w_t L_t$). From this *static and unconstrained* problem, derive the First Order Conditions, and show how the marginal productivity of a factor equals its cost.

2) Suppose now that the technology shows constant returns to scale, and in particular a Cobb-Douglas function: $F(K_t, L_t) = (K_t)^{\alpha}(L_t)^{1-\alpha}$, so that $Y_t = Z_t (K_t)^{\alpha}(L_t)^{1-\alpha}$.
a) Show how the Cobb-Douglas technology has constant returns to scale.
b) Plug the optimality conditions in the objective function (the profits), and show that the profits are zero for this representative firm.

3) Take a step back to the firm's problem. Suppose output/value added Y is a general function of TFP, labor, and capital, i.e. $Y = ZF(L, K)$. Assuming perfect competition and constant returns to scale, derive the following expression for logged TFP: $z = y - (1-\alpha)\ell - \alpha k$, where lower-case letters denote logs of their level counterparts and $1-\alpha$ the wage bill as a share of total revenues (wL/Y).

Hint: Follow the procedure: (i) take the total derivative of $Y = f(Z, L, K) = ZF(L, K)$ w.r.t. each factor, (ii) divide by total quantity Y and multiply/divide by factor quantities in each quantity, (iii) use the assumption of perfect competition – where firms are minimizing costs and factor prices p_i of input q_i equal their marginal products $p_i = \frac{df}{dq_i}\frac{C}{Y}\mu$, where $\mu = \frac{\partial C}{\partial Y}\frac{Y}{C}$ is the cost elasticity of quantity produced (scale elasticity) – hence $\frac{df}{dq_i} = \frac{p_i}{\mu}\frac{Y}{C}$ (iv) use the assumption of constant returns to scale, where the scale elasticity $\mu = 1$, and (v) integrate the log differences $\int d\log x = \log x^*$, where the integration constant is chosen arbitrarily.

3.1) What does this approximation tell us about Cobb-Douglas production functions?
3.2) Is it a good way of estimating productivity? Are those assumptions realistic?

2 The Household problem (15 pts) Lagrangian method

Now, let us consider the Household problem:

$$\max_{\{C_t, A_{t+1}, L_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$

$$\text{s.t.} \quad C_t + A_{t+1} \leq (1+r_t)A_t + w_t L_t \qquad \forall\, t \geq 0 \qquad (A_0 \text{ given})$$

$$\text{and} \quad \lim_{T\to\infty} \frac{A_{T+1}}{\prod_{s=1}^{T+1}(1+r_s)} \geq 0$$

A representative consumer supplies labor L_t at the wage w_t, which is time-varying but deterministic (its fluctuations are perfectly anticipated), consumes a good C_t, and saves using an asset A. In period t, the consumer owns a quantity A_t of savings (when $A_t \leq 0$ this saving is in fact a debt) and decides how much to save for the new period, A_{t+1}. The second constraint is the no-Ponzi condition of Section 0.3, without which the problem has no solution.

Note that we could relabel this budget constraint as $C_t + Q_t \tilde{A}_{t+1} = \tilde{A}_t + w_t L_t$, where the one-period asset $\tilde{A}_{t+1}$ is bought at price $Q_t = \frac{1}{1+r_{t+1}}$ and "returns" 1 next period, with $\tilde{A}_t = (1+r_t)A_t$. This convention will be used in later sections.

(a.) Apply the KKT theorem of Section 0.3 to solve the Household problem (applying a theorem means checking explicitly its different hypotheses). The household maximizes over its consumption path, its saving path and its labor supply path. There are three choice variables, hence three first-order conditions; eliminating the Lagrange multipliers leaves two optimality conditions:

(i) an Euler equation, relating present and future marginal utility of consumption;

(ii) a labor-consumption tradeoff, relating the marginal utility of consumption to the marginal disutility of labor.

(b.) Apply these conditions to the special functional form of utility: separable between labor and consumption, CRRA in consumption and with a constant Frisch-elasticity: $U(C_t, L_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi}$, where σ is the risk aversion coefficient – which is also the *inverse* of the intertemporal elasticity of substitution (IES) – and φ is the inverse of the Frisch elasticity of labor supply.

- Explain these relations with economic intuitions.
- How does the consumption path change with r_t? How do consumption and saving react when r_t changes, in the case where the IES is low ($\sigma > 1$) vs. when it is high ($\sigma < 1$)?
- Lastly, expressing $Q_t = \frac{1}{1+r_{t+1}}$ as a function of the other variables, how is the asset priced (what is Q_t) if $C_{t+1} \gg C_t$?

3 The Household problem (15 pts) Dynamic programming

Now, let us consider the Household problem:

$$\max_{\{C_t, A_{t+1}, L_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$
$$\text{s.t.} \qquad C_t + A_{t+1} \leq (1 + r_t)A_t + w_t L_t \qquad \forall\, t \geq 0 \qquad (A_0 \text{ given})$$

This is exactly the same problem as in the previous section – with the same no-Ponzi condition – but we now solve it ***recursively***, using the "Dynamic programming" method. Again, the household maximizes over the consumption path and the labor supply path, and you should obtain the very same two types of optimality conditions:
(i) an Euler equation and (ii) a labor-consumption tradeoff. The point of the exercise is the method, not the result: the two approaches must agree, and the recursive one is the only one that will still work once we introduce risk (in Section 8).

The idea here is to solve[6] the "Bellman" equation: the objective can be rewritten as an iterative "Value function":

$$V_t(A_t) = \max_{\{C_j, A_{j+1}, L_j\}_{j \geq t}} \left[\sum_{j=t}^{\infty} \beta^{j-t} U(C_j, L_j) \right]$$
$$\Leftrightarrow V_t(A_t) = \max_{C_t, A_{t+1}, L_t} \Big[U(C_t, L_t) + \beta V_{t+1}(A_{t+1}) \Big]$$

To solve this problem, you need to derive (i) the KKT first-order conditions for this new (iterative) maximization problem[7], and (ii) a new "Envelope condition". This second condition gives the derivative of the value function (w.r.t. A_t) and, through the chain rule, is linked to the "Envelope Theorem". In this theorem[8], when a function is maximized over a variable (V_t is maximized over A_{t+1} here) and hence optimal in this sense, its derivative w.r.t. that same variable is null.
Combining the different equations and manipulating the terms, you should obtain (i) the Euler equation and (ii) the labor-consumption tradeoff (exactly as before!).

Apply these conditions to the special functional form of utility: CRRA in consumption and with a constant Frisch-elasticity: $U(C_t, L_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi}$

[6]During the course, we will discuss the general method to solve such problems (and prove that the sequential problem and its recursive formulation are equivalent).
[7]Note that we are back to the first "static" version of the KKT theorem (Section 0.1).
[8]We will come back to this theorem during the course as well.

4 The New-Keynesian model (10 pts) Nominal rigidities, IS-curve, Phillips curve and monetary policy

1. [Optional question] To solve the New Keynesian model, one of the steps is to solve the following Household problem:

$$\max_{\{C_t, B_{t+1}, L_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$

$$\text{s.t.} \qquad C_t P_t + Q_t B_{t+1} \leq B_t + W_t L_t + T_t$$

Here P_t is the price level, W_t the nominal wage, B_{t+1} a one-period nominal bond bought at price Q_t and paying one unit of currency at $t+1$, and T_t lump-sum transfers (including firms' profits). This problem is really similar to the one of Section 2 (again everything is deterministic for the sake of simplicity). Using the given functional form $U(C_t, L_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi}$, derive [or simply state] the Euler equation and the labor-consumption tradeoff, using the method of your choice (the Lagrangian method or the Dynamic programming method)[9].
Briefly explain what changes in the Euler equation and in the labor-consumption tradeoff, for instance when inflation changes the optimal decisions.

2. ***[Compulsory question]*** Recall the log-linearized formulas[10] for (a) the dynamic IS-equation (linking the present and future output gaps and interest rate), (b) the New-Keynesian Phillips Curve (linking present and future inflation rates and aggregate demand) and (c) the interest rate rule[11] (showing how monetary policy evolves). Describe these relations and their economic intuitions.
For the monetary policy rule, briefly explain the Taylor principle.

[9] This exercise is advised for those who feel the need to practice.

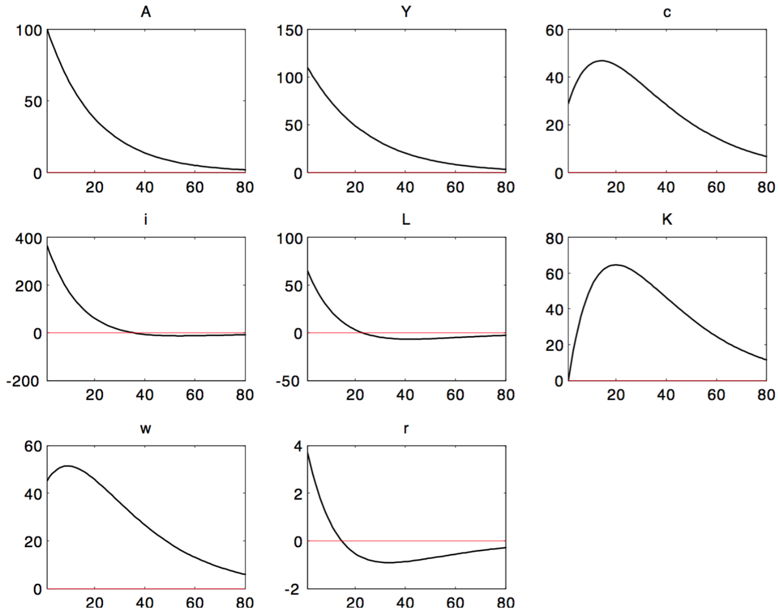
[10] Simply state the three equations, no need to derive them! The main textbook for this type of model is: J. Gali - Monetary Policy, Inflation and the Business cycle

[11] There are different types of monetary policy rule, choose one and explain it.

5 The dynamics of the two models (12 pts) Impulse response functions (IRF) to a technology shock

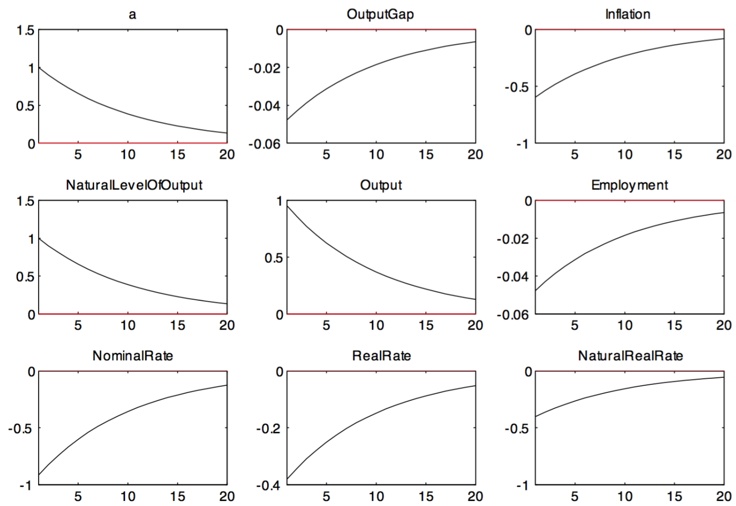
This is the response of the two standard models to a technology shock.[12]

Figure 1: *Baseline RBC model*



The relations structuring the first (RBC) model are the Euler equation, the labor-consumption tradeoff, the definition of output (Cobb-Douglas technology, and $Y_t = C_t + I_t$), the law of motion of capital ($K_{t+1} = (1-\delta)K_t + I_t$), the factor prices w_t and r_t set equal to the marginal productivities of the factors, and an AR(1) process for TFP.

Figure 2: *Baseline NK model*



[12] *Notation in the figures.* The plots were produced in Dynare/Matlab, with notations differing slightly from the ones used previously. In the *RBC* figure: "A" is the TFP Z_t, "Y" is output Y_t, "c" is consumption C_t, "i" is investment I_t, "L" is labor L_t, "K" is capital K_t, "w" is the real wage w_t and "r" is the interest rate r_t. In the *NK* figure: "a" is the (log) TFP z_t, "NaturalLevelOfOutput" is y_t^n, "Output" is y_t, "OutputGap" is the output gap $\widetilde{y}_t = y_t - y_t^n$, "Inflation" is π_t, "Employment" is $\hat{\ell}_t$, "NominalRate" is i_t, "RealRate" is the ex-ante real rate $i_t - \mathbb{E}_t(\pi_{t+1})$ and "NaturalRealRate" is r_t^n. In both figures, all variables are plotted as deviations from their steady-state value, and the horizontal axis is the number of periods after the shock.

The relations structuring the second (NK) model are the IS equation (or Euler equation), the NK-Phillips Curve, the monetary policy rule for the nominal interest rate, the definitions of the natural level of output and of the natural rate of interest (the ones that would prevail in the flexible price economy), the output gap (effective output minus natural output), the real rate as the nominal rate net of expected inflation, and employment.

Describe the economic mechanisms underlying the two sets of Impulse Response Functions (IRF) following a technology shock, first in the Real-Business-Cycle model, and second in the New-Keynesian model.

6 Stochastic income processes (30 pts) Random walk, Markov chain, Martingale, AR(1), e tutti quanti

The first half of this problem set was written in *general equilibrium* and in a *deterministic* environment: households and firms interacted, and Y_t denoted aggregate output. In this second half, we zoom in on a single household in *partial equilibrium* – prices and the income process are taken as given – and we introduce *risk*. Accordingly, the notation shifts: from now on Y_t denotes the household's exogenous *income* ($\sim$ labor income $w_t L_t$, now random, and labor supply is no longer chosen by agents). This is the notation used in Sections 7 and 8, and in the consumption-saving and heterogeneous-agent literature more broadly.

We now introduce a simple shock for income $Y_t = Y(s_t)$. It fluctuates randomly according to a stochastic process, where s_t is a "state of the world" (or alea) at date t that belongs to the information set $\mathcal{I}_t$ (what is known to be possible to happen at date t)[13]. More specifically, a sequence of state-of-the-world $\{s_0, s_1, \ldots, s_{t-1}, s_t\} = s^t$ defines a "history" s^t of shocks until t (this is the economist way of writing things compactly).

Putting some structure on this process $Y(s^t)$ (making it depend on the past), we consider an Autoregressive – AR(1) – process for income:

$$\begin{cases} Y_t(s^t) & = \rho_Y Y_{t-1}(s^{t-1}) + \varepsilon(s_t) \\ Y_{t_0} & = y \end{cases}$$

where $\rho_Y \in \mathbb{R}$ is a parameter and ε is an i.i.d. random variable (i.e. a function of the state of the world), that simply[14] takes three values: $+1$ with proba p, -1 with proba q and 0 with proba $1-p-q$. The following questions (b.) to (g.) are independent from each other (but most of them need the answer of (a.)).

(a.) Compute the conditional expectation $\mathbb{E}(Y_t(s^t)|\mathcal{I}_{t-1})$, as a function of the parameters and note that this expectation is a function of a variable depending on s^{t-1}.
- Using the same logic, write the value $\mathbb{P}(Y_t(s^t) = z|\mathcal{I}_{t-1})$ (Hint: you may need to use indicator functions $\mathbb{1}\{W = w\}$ for a random variables W and values w to be determined).

(b.) Show what are the conditions on the parameters (ρ_Y, p and q, etc.[15]), for Y_t to be a martingale (or a sub-martingale, or a super-martingale).

(c.) Does this stochastic process respect the Markov property? Can we say that it is a Markov chain (i.e. Markov process)? What is the set of states? (Hint: this needs not be finite). Write the transition matrix as function of the parameters, for $\rho_Y = 1$.

(d.) Assume that $\rho_Y = 1$. This process is called a random walk. Suppose that there are many agents with the same income process, starting from $y \in \mathbb{Z}$. What is the law of motion of

[13] In measure theory language, this information set is the $\sigma-$algebra generated by the stochastic process until t (and $\{\mathcal{I}_t\}_t$ is a filtration: $\mathcal{I}_{t-1} \subset \mathcal{I}_t \subset \mathcal{I}_{t+1}$), i.e. it implies that $Y(s_t)$ is measurable with respect to this $\sigma-$algebra: it doesn't include values that are "outside" this set (if measure theory is too abstract for you, don't worry and just ignore this last sentence/footnote!)

[14] If you find that too easy, feel free to consider that as a random variable $\varepsilon(s_t) \sim \mathcal{N}(\mu_Y, \sigma_Y)$

[15] This includes μ_Y if you consider a Normal law for ε

the distribution π_t over the income "states"?
- Give explicitly the distribution of income at dates $t = 1$ and $t = 2$, when $y = 2$.
- Do you expect this process to have a stationary distribution over income states? (if yes, which one? if not, why not?)

(e.) Assume that $\rho_Y < 1$. Write the expression of $Y(s^t)$ as a function of past shocks and parameters.
- In the VAR language, rewrite this with lag operators. Moreover, if the first date was $t_0 \to -\infty$, can you reexpress this process as a MA(∞) process? As a consequence, can you rewrite this lag polynomial in a more compact way? (Hint: try to "inverting" an operator "as if" it was a standard division and use the rule for geometric series).
- In this setting, (or in the case where $t_0 = 0$ and $y = 0$) can you compute the variance $\mathbb{V}\text{ar}(Y_t)$ and the covariance $\mathbb{C}\text{ov}(Y_t, Y_{t-1})$? (Hint: recall that $\mathbb{V}\text{ar}(Z_t) = \mathbb{E}(Z_t^2) - \mathbb{E}(Z_t)^2$, for random variables Z_t that admit a finite second moment).
- Using the answer for the last subquestion, can you propose a method of moments (GMM) to estimate p, q and ρ_Y.

(f.) Assume $\varepsilon \sim \mathcal{N}(\mu_Y, \sigma_Y)$ for this question. Suppose you had data on $Y_0, Y_1, \dots, Y_T$, and you would like to estimate the parameters ρ_Y, μ_Y and σ_Y using Maximum Likelihood estimation (MLE). Write the (log-)likelihood, i.e. the objective of this maximization problem.

(g.) Suppose ε is again the original three states $\{+1, 0, -1\}$ random variable, with probabilities p, q such that $\mathbb{E}(\varepsilon) = 0$ and $\mathbb{V}\text{ar}(\varepsilon) = 1$ (what are these values of p and q?). Assume that $Y_0 = 0$ and that the process Y_t is both a martingale and a Markov chain (i.e. we have the properties of questions (b.) and (c.)).
- What do you expect this process to look like when the time period becomes infinitesimally small, i.e. $\Delta t = t_{i+1} - t_i \to 0$. Can you give the name of this stochastic process?
- More formally, define the rescaled process

$$W^{(n)}(t) = \frac{Y_{\lfloor nt \rfloor}}{\sqrt{n}}, \qquad t \in [0, 1]$$

Can you determine if $W^{(n)}(1)$ converge in distribution toward a specific random variable? If yes, which one? What is the name of this theorem?
- (Advanced) Can one show this convergence for every t, and hence determine the law of $W(t)$ at the limit for n? What is the name of this theorem?

7 The Household problem in Arrow-Debreu economies (10 pts) State-dependent consumption & portfolio choice

Using a stochastic process for income, we assume that the market is complete in the Arrow-Debreu (AD) sense: for each history s^t and each state of the world s_{t+1}, there exists an asset $A_{t+1}(s_{t+1}|s^t)$ that yields one unit of consumption in state s_{t+1}, and that can be purchased at date t after history s^t, at the price $Q_t(s_{t+1}|s^t)$ (for compact notation: $A_{t+1}(s_{t+1}|s^t) \equiv A_{t+1}(s^{t+1})$ and $Q_t(s_{t+1}|s^t) \equiv Q_t(s^{t+1})$). Now, given this complete market structure, the household maximizes its expected utility, given the probability $\pi(s^t)$ of history s^t.

$$\max_{\{C_t(s^t)\}_0^\infty, \{A_{t+1}(s^{t+1})\}_0^\infty} \sum_{t=0}^{\infty} \sum_{s^t \subset \mathcal{I}_t} \beta^t \pi(s^t) U\big(C_t(s^t)\big)$$

$$\text{s.t.} \qquad C_t(s^t) + \sum_{s_{t+1}} Q_t(s_{t+1}|s^t) A_{t+1}(s_{t+1}|s^t) \leq A_t(s^t) + Y_t(s^t) \qquad \forall\, t \geq 0,\ \forall\, s^t$$

with $A_0(s^0)$ given, and a no-Ponzi condition holding state by state, as in Section 0.3. Using the Lagrangian method, derive the Euler equation and specify the stochastic discount factor.

8 The Household problem in incomplete markets (20 pts) Self-insurance and Dynamic programming

Consider the same problem as in the previous exercise. This time the market is incomplete: there is no asset contingent on each state of the world s_{t+1}. The only asset is a "risk-free bond"

$$\max_{\{C_t(s^t)\}_0^\infty, \{A_{t+1}(s^t)\}_0^\infty} \sum_{t=0}^{\infty} \sum_{s^t \subset \mathcal{I}_t} \pi(s^t) \beta^t U\big(C_t(s^t)\big)$$

$$\text{s.t.} \qquad C_t(s^t) + Q_t(s^t) A_{t+1}(s^t) \leq A_t(s^t) + Y_t(s^t) \qquad \forall\, t \geq 0,\ \forall\, s^t$$

with $A_0(s^0)$ given and a no-Ponzi condition, as in Section 0.3. To alleviate notations, we drop the history s^t where no confusion arises: all the variables chosen at date t are measurable w.r.t. $\mathcal{I}_t$, i.e. they may depend on the whole history s^t but not on the shocks s_{t+1} still to come.

For that, we will use a "Dynamic programming" procedure. Again, the household maximizes over its consumption and saving paths, and we will obtain an Euler equation.

The idea here is to solve[16] the "Bellman" equation: the objective can be rewritten as an iterative "Value function":

$$\begin{aligned}
V_t(A_t, Y_t) &= \max_{\{C_j, A_{j+1}\}_{j \geq t}} \mathbb{E}\Big[\sum_{j=t}^{\infty} \beta^{j-t} U(C_j) \big| \mathcal{I}_t\Big] \\
&= \max_{\{C_j, A_{j+1}\}_{j \geq t}} \sum_{j=t}^{\infty} \sum_{s^j \subset \mathcal{I}_j} \pi(s^j | s^t) \beta^{j-t} U(C_j) \\
\Leftrightarrow \qquad V_t(A_t, Y_t) &= \max_{C_t, A_{t+1}} \Big[U(C_t) + \beta\, \mathbb{E}\big(V_{t+1}(A_{t+1}, Y_{t+1}) \big| \mathcal{I}_t\big)\Big] \\
&= \max_{C_t, A_{t+1}} \Big[U(C_t) + \beta \sum_{s_{t+1} \subset \mathcal{I}_{t+1}} \pi(s_{t+1} | s^t)\, V_{t+1}\big(A_{t+1}, Y_{t+1}(s_{t+1})\big)\Big]
\end{aligned}$$

To solve this problem, you need to derive
(i) the KKT first-order conditions for this new (iterative) maximization problem[17], and
(ii) a new "Envelope condition". This second condition gives the derivative of the value function (w.r.t. A_t) and, through the chain rule, is linked to the "Envelope Theorem". In this theorem, when a function is maximized over a variable (V_t is maximized over A_{t+1} here) and hence optimal in this sense, its derivative w.r.t. that same variable is null.

Combining the different equations and manipulating the terms, show the Euler equation and show the similarities and differences with the Arrow-Debreu/complete market economy.

[16] During the course, we will discuss the general method to solve such problems (and prove that the sequential problem and its recursive formulation are equivalent).

[17] Note that we are back to the first "static" version of the KKT theorem (Section 0.1).