Uncertainty and the Inequality of Climate Change

WORK IN PROGRESS

Thomas Bourany
THE UNIVERSITY OF CHICAGO

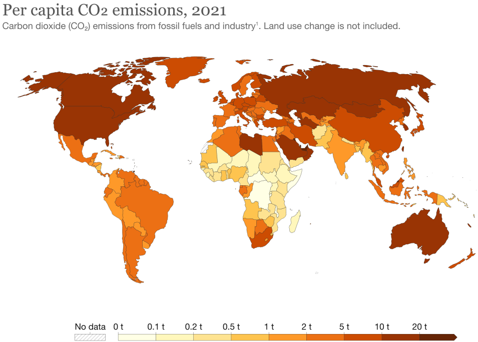
Capital Theory

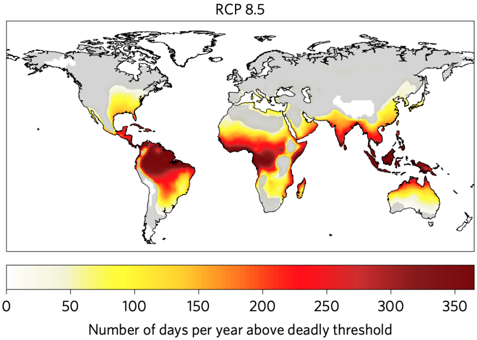
Mar 2023

Introduction – Motivation

Global warming is caused by greenhouse gas emissions (GHG) generated by human economic activity :

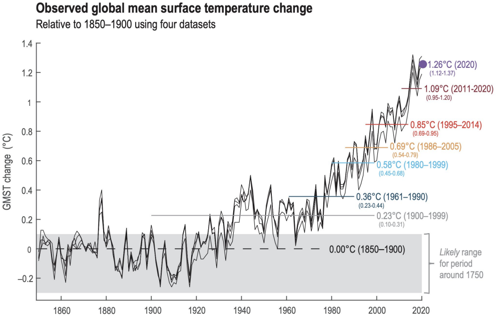
- ***Unequal causes :*** Developed economies account for over 65% of cumulative GHG emissions ($\sim$ 25% each for the EU and the US)
- ***Unequal consequences :*** Increase in temperatures disproportionately affects developing countries where the climate is already warm

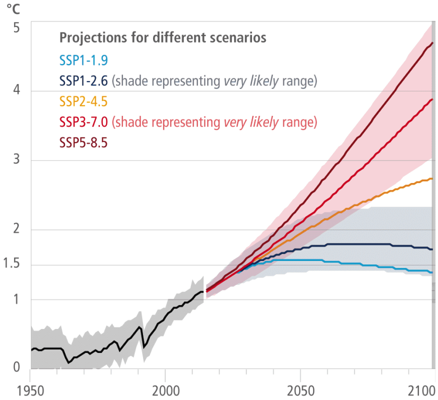




Introduction – Uncertainty

- However, the impact of climate change is uncertain for several reasons :
  - (i) Climate forecasts : temperature trajectories for a given path of emissions
  - (ii) Future growth : levels of future output for given damages
  - (iii) Path of emissions : Likelihood that pledges/mitigation policies will be implemented.





Introduction – this project

- Which countries will be the most affected by climate change and climate risk ?
  - Is the price of carbon heterogeneous across regions ? and why ?
  - Is the impact of climate risk quantitatively important ?

Introduction – this project

- Which countries will be the most affected by climate change and climate risk ?
  - Is the price of carbon heterogeneous across regions ? and why ?
  - Is the impact of climate risk quantitatively important ?
- Develop a simple and flexible model of climate economics
  - Standard Neoclassical – IAM model with heterogeneous regions

Introduction – this project

- Which countries will be the most affected by climate change and climate risk ?
  - Is the price of carbon heterogeneous across regions ? and why ?
  - Is the impact of climate risk quantitatively important ?
- Develop a simple and flexible model of climate economics
  - Standard Neoclassical – IAM model with heterogeneous regions
- Provide a new numerical methodology to :
  - Simulate globally – and sequentially – models with heterogeneous agents/countries
  - Handle aggregate shocks and different trajectories of temperatures

Introduction – this project

- Which countries will be the most affected by climate change and climate risk ?
  - Is the price of carbon heterogeneous across regions ? and why ?
  - Is the impact of climate risk quantitatively important ?
- Develop a simple and flexible model of climate economics
  - Standard Neoclassical – IAM model with heterogeneous regions
- Provide a new numerical methodology to :
  - Simulate globally – and sequentially – models with heterogeneous agents/countries
  - Handle aggregate shocks and different trajectories of temperatures
- Evaluate the heterogeneous welfare costs of global warming
  - Local Social Cost of Carbon can vary tenfold across countries
  - ... and $> 50\%$ across states of the world (within countries).

Introduction – related literature

- Classic Integrated Assessment models (IAM), without or with country heterogeneity :
  - Nordhaus' Multi-regions DICE (2016), Golosov Hassler Krusell Tsyvinski (2014), Dietz van der Ploeg, Rezai, Venmans (2021), among others
  - Kotlikoff, Kubler, Polbin, Scheidegger (2021), Hassler, Krusell, Olovsson, Smith (2019-2021), Cruz, Rossi-Hansberg (2022), Rudik et al (2022)
  - *This project :* Studies uncertainty with heterogeneity
- Climate models with risk & uncertainty :
  - Cai, Lontzek, Judd (2019), Barnett, Brock, Hansen (2022), Bilal, Rossi-Hansberg (2023)
  - *This project :* Includes heterogeneity and redistribution effects of climate & carbon taxation
- Heterogeneous Agents models with Aggregate Risk
  - Krusell-Smith (1998), Bhandari, Evans, Golosov, Evans (2018-), Proehl (2020), Schaab (2020), Fernandez-Villaverde et al (2022), Bilal (2022)
    Proba. approach to MFG : Carmona, Delarue (2018) and many more
  - *This project :* Studies climate externalities and Pigouvian taxation

Model

- Neoclassical economy, in continuous time
  - countries/regions $i \in \mathbb{I}$: ex-ante heterogeneous : productivity z_i
  - ex-post heterogeneity in capital and temperature $x_i = \{k_i, \tau_i\}$
  - Aggregate variable : carbon $\mathcal{S}$tock in atmosphere : $\mathcal{X} = \{\mathcal{S}\}$
- Household problem in country i :

$$\max_{\{c_{i,t}, e^f_{i,t}, e^r_{i,t}\}} \int_{t_0}^{\infty} e^{-\rho t} \, u(c_{i,t}) dt$$

- Dynamics of capital in every country i :

$$dk_{i,t} = \Big(\mathcal{D}^y(\tau_{i,t}) z_{i,t} f(k_{i,t}, e^f_{i,t}, e^r_{i,t}) - \bar{\delta} k_{i,t} - q^f_t e^f_{i,t} - q^r_t e^r_{i,t} - c_{i,t}\Big) dt$$

  - Damage function $\mathcal{D}^y(\tau_t)$ affect country's production
  - Energy mix : fossil e^f_t – emitting carbon – vs. renewable e^r_t
  - Prices, fossil q^f_t and non-carbon q^r_t, exogenous (energy firms with linear production fct)

Climate model :

- Fossil energy input e_t^f causes climate externality

$$\mathcal{E}_t = \int_{\mathbb{I}} \xi \, e_{i,t}^f \, di$$

- World climate – cumulative GHG in atmosphere $\mathcal{S}_t$ leads to increase in temperature

$$d\mathcal{S}_t = \left(\mathcal{E}_t - \delta_s \mathcal{S}_t\right) dt$$

- Impact of climate on country's local temperature :

$$d\tau_{i,t} = \zeta(\Delta_i \chi \mathcal{S}_t - \tau_{i,t})dt + \Delta_i \sigma dB_t^0$$

- Aggregate risk σdB_t^0
- Simple model :
  - Climate sensitivity to carbon χ, Climate reaction/inertia ζ, Carbon content of fossils ξ, Country scaling factor Δ_i, Carbon exit for atmosphere δ_s
  - Possibility of a more detailed Climate model : Detailed climate model

Model Solution :

- Global method :
  - Aggregate risk pushes the equilibrium far away from steady state
  - Impossibility to use first/second order Taylor (local) approximations
- Sequential approach
  - Relying on Pontryagin Maximum Principle (PMP)
  - Extension to the stochastic case and mean-field / heterogeneous agents
- Numerical method :
  - Shooting algorithm
- Possibility to handle Optimal Policy and Ramsey Problem Social Planner

Model solution – general formulation

- States variables : Even more general formulation
  - Individual : $x_{i,t} \in \mathbb{X} \subset \mathbb{R}^d$ (possibly with state-constraints), with distribution $P_{x,t}$
  - Aggregate : $\mathcal{X}_t \in \overline{\mathbb{X}} \subset \mathbb{R}^d$, and controls $c^{\star}(\cdot) \in \mathbb{C}$

$$dx_{i,t} = b(x_{i,t}, \mathcal{X}_t, c^{\star}_{i,t})dt + \sigma dB^0_t$$
$$d\mathcal{X}_t = \bar{b}(\mathcal{X}_t, P_{x,t})dt + \bar{\sigma} dB^0_t$$

Model solution – general formulation

- States variables : Even more general formulation
  - Individual : $x_{i,t} \in \mathbb{X} \subset \mathbb{R}^d$ (possibly with state-constraints), with distribution $P_{x,t}$
  - Aggregate : $\mathcal{X}_t \in \overline{\mathbb{X}} \subset \mathbb{R}^d$, and controls $c^\star(\cdot) \in \mathbb{C}$

$$dx_{i,t} = b(x_{i,t}, \mathcal{X}_t, c^\star_{i,t})dt + \sigma dB^0_t$$
$$d\mathcal{X}_t = \bar{b}(\mathcal{X}_t, P_{x,t})dt + \bar{\sigma} dB^0_t$$

- Hamiltonian :

$$\mathcal{H}(x, y, \mathcal{X}, \mathcal{Y}) = \max_{c \in \mathbb{C}} \big(u(x, c) + b(x, \mathcal{X}, c) \cdot y\big) + \bar{b}(\mathcal{X}_t, P_{x,t}) \cdot \mathcal{Y}$$

- Optimal control $c^\star \in \text{argmax}_{c \in \mathbb{C}} \big(u(x, c) + b(x, \mathcal{X}, c) \cdot y\big)$

Model solution – general formulation

- States variables : Even more general formulation
  - Individual : $x_{i,t} \in \mathbb{X} \subset \mathbb{R}^d$ (possibly with state-constraints), with distribution $P_{x,t}$
  - Aggregate : $\mathcal{X}_t \in \overline{\mathbb{X}} \subset \mathbb{R}^d$, and controls $c^\star(\cdot) \in \mathbb{C}$

$$dx_{i,t} = b(x_{i,t}, \mathcal{X}_t, c^\star_{i,t})dt + \sigma dB^0_t$$
$$d\mathcal{X}_t = \bar{b}(\mathcal{X}_t, P_{x,t})dt + \bar{\sigma} dB^0_t$$

- Hamiltonian :

$$\mathcal{H}(x, y, \mathcal{X}, \mathcal{Y}) = \max_{c \in \mathbb{C}} \Big(u(x, c) + b(x, \mathcal{X}, c) \cdot y \Big) + \bar{b}(\mathcal{X}_t, P_{x,t}) \cdot \mathcal{Y}$$

- Optimal control $c^\star \in \mathrm{argmax}_{c \in \mathbb{C}} \big(u(x, c) + b(x, \mathcal{X}, c) \cdot y \big)$
- Using the Stochastic PMP :

$$dy_{i,t} = -D_x \mathcal{H}(x_{i,t}, y_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t})dt + z_{i,t} dB^0_t$$
$$d\mathcal{Y}_{i,t} = -D_{\mathcal{X}} \mathcal{H}(x_{i,t}, y_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t})dt + \mathcal{Z}_{i,t} dB^0_t$$

Application to climate models

- States $x_{i,t} = (k_{i,t}, \tau_{i,t})$ and $\mathcal{X}_t = \mathcal{S}_t$
- Costates $(y_{i,t}, \mathcal{Y}_{i,t}) = (\lambda^k_{i,t}, \lambda^\tau_{i,t}, \lambda^S_{i,t})$
- Controls $c_{i,t} = (c_{i,t}, e^f_{i,t}, e^r_{i,t})$

$$\mathcal{H}_i(x, \mathcal{X}, \{c, e^f, e^r\}, \{\lambda^k, \lambda^\tau, \lambda^S\}) = u(c_i) + \lambda^k_{i,t}\Big(\overbrace{\mathcal{D}(\tau_{it})f(k_t, e^f_{it}, e^r_{it}) - \bar{\delta}k_t - q^f_t e^f_{it} - q^r_{it}e^r_{it} - c_t}^{=b_1(x_{i,t},\mathcal{X}_i,c^\star_{i,t})}\Big)$$
$$+ \lambda^\tau_{i,t}\overbrace{\zeta\Big(\Delta_i\,\chi\,\mathcal{S}_t - (\tau_{it} - \tau_{i0})\Big)}^{b_2(x_{i,t},\mathcal{X}_i,c^\star_{i,t})} + \lambda^S_{i,t}\Big(\overbrace{\mathcal{E}_t - \delta^s\mathcal{S}_t}^{=\bar{b}(P_x,\mathcal{X}_{i,t})}\Big)$$

- Optimal controls :

$$\lambda^k_{i,t} = u'(c_{i,t}) \qquad q^f_t = MPe^f_{it} \qquad q^r_{it} = MPe^r_{it}$$

- Dynamics of costates
  - More details : More details PMP

Model solution : FBSDE system for MFG systems

- Coupled FBSDE system for each agent

$$\begin{cases} dx_{i,t} &= D_y \mathcal{H}(x_{i,t}, y_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}) dt + \sigma dB_t^0 \\ dy_{i,t} &= -D_x \mathcal{H}(x_{i,t}, y_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}) dt + z_{i,t} dB_t^0 \end{cases}$$

- Question : What else do we need ?

Model solution : FBSDE system for MFG systems

- Coupled FBSDE system for each agent

$$\begin{cases} dx_{i,t} &= D_y\mathcal{H}(x_{i,t},y_{i,t},\mathcal{X}_t,\mathcal{Y}_{i,t})dt + \sigma dB_t^0 \\ dy_{i,t} &= -D_x\mathcal{H}(x_{i,t},y_{i,t},\mathcal{X}_t,\mathcal{Y}_{i,t})dt + z_{i,t}dB_t^0 \end{cases}$$

- Question : What else do we need ?
  - The initial condition y_0

Model solution : FBSDE system for MFG systems

- Coupled FBSDE system for each agent

$$\begin{cases} dx_{i,t} & = D_y \mathcal{H}(x_{i,t}, y_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}) dt + \sigma dB_t^0 \\ dy_{i,t} & = -D_x \mathcal{H}(x_{i,t}, y_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}) dt + z_{i,t} dB_t^0 \end{cases}$$

- Question : What else do we need ?
  - The initial condition y_0
  - A boundary condition on y_T or transversality $\lim_{t \to \infty} e^{-\rho t} x_t y_t = 0$

Model solution : FBSDE system for MFG systems

- Coupled FBSDE system for each agent

$$\begin{cases} dx_{i,t} & = D_y\mathcal{H}(x_{i,t},y_{i,t},\mathcal{X}_t,\mathcal{Y}_{i,t})dt + \sigma dB_t^0 \\ dy_{i,t} & = -D_x\mathcal{H}(x_{i,t},y_{i,t},\mathcal{X}_t,\mathcal{Y}_{i,t})dt + z_{i,t}dB_t^0 \end{cases}$$

- Question : What else do we need ?
  - The initial condition y_0
  - A boundary condition on y_T or transversality $\lim_{t\to\infty} e^{-\rho t}x_t y_t = 0$
  - The individual risk loading in the costate $z_{i,t}$:
    - Expectation error in the law of motion of $y_{i,t}$

$$z_{i,t} = \mathbb{E}^{\epsilon}\left[\frac{y_{i,t+dt}(\epsilon) - y_{i,t} + D_x\mathcal{H}(x_{i,t},y_{i,t},\mathcal{X}_t,\mathcal{Y}_{i,t})dt}{dB_t^0}\right]$$

Model solution : FBSDE system for MFG systems

- Coupled FBSDE system for each agent

$$\begin{cases} dx_{i,t} &= D_y \mathcal{H}(x_{i,t}, y_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}) dt + \sigma dB_t^0 \\ dy_{i,t} &= -D_x \mathcal{H}(x_{i,t}, y_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}) dt + z_{i,t} dB_t^0 \end{cases}$$

- Question : What else do we need ?
  - The initial condition y_0
  - A boundary condition on y_T or transversality $\lim_{t\to\infty} e^{-\rho t} x_t y_t = 0$
  - The individual risk loading in the costate $z_{i,t}$:
    - Expectation error in the law of motion of $y_{i,t}$

$$z_{i,t} = \mathbb{E}^{\epsilon}\left[\frac{y_{i,t+dt}(\epsilon) - y_{i,t} + D_x \mathcal{H}(x_{i,t}, y_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}) dt}{dB_t^0}\right]$$

    - BSDE theory : keep the co-state measurable w.r.t. dB_t^0, despite running backward.
      $\Rightarrow$ ***Intuition :*** even if agents are forward-looking, they can't know the future.
    - Advantage : Numerically Feasible via Monte Carlo or Tree Methods

Method – Shooting algorithm

- Deterministic case / Representative agent :
  1. Start from initial condition X_{t_0} and the guess Y_{t_0}
  2. Simulate the sequence (X_t, Y_t) for $t \in [t_0, T]$ using the forward ODE system + finite diff.
  3. Update the guess Y_{t_0} to match the terminal condition Y_T
     - In practice, simulate the backward $\tilde{Y}_t$ for $t \in [t_0, T]$ and minimize $\int_{t_0}^{T} (\tilde{Y}_t - Y_t)^2 dt$

Method – Shooting algorithm

- Deterministic case / Representative agent :
  1. Start from initial condition X_{t_0} and the guess Y_{t_0}
  2. Simulate the sequence (X_t, Y_t) for $t \in [t_0, T]$ using the forward ODE system + finite diff.
  3. Update the guess Y_{t_0} to match the terminal condition Y_T
     - In practice, simulate the backward $\tilde{Y}_t$ for $t \in [t_0, T]$ and minimize $\int_{t_0}^{T} (\tilde{Y}_t - Y_t)^2 dt$

- Heterogeneity : the same method works !
  - Difficulty : guess y_{i,t_0} for the (large ?) set of agents, $\forall i \in \mathbb{I}$
  - Leverage optimization routines (and automatic differentiation)
  - Could couple that with Monte Carlo and Deep Learning

Method – Shooting algorithm

- Deterministic case / Representative agent :
  1. Start from initial condition X_{t_0} and the guess Y_{t_0}
  2. Simulate the sequence (X_t, Y_t) for $t \in [t_0, T]$ using the forward ODE system + finite diff.
  3. Update the guess Y_{t_0} to match the terminal condition Y_T
     - In practice, simulate the backward $\tilde{Y}_t$ for $t \in [t_0, T]$ and minimize $\int_{t_0}^{T} (\tilde{Y}_t - Y_t)^2 dt$

- Heterogeneity : the same method works !
  - Difficulty : guess y_{i,t_0} for the (large ?) set of agents, $\forall i \in \mathbb{I}$
  - Leverage optimization routines (and automatic differentiation)
  - Could couple that with Monte Carlo and Deep Learning

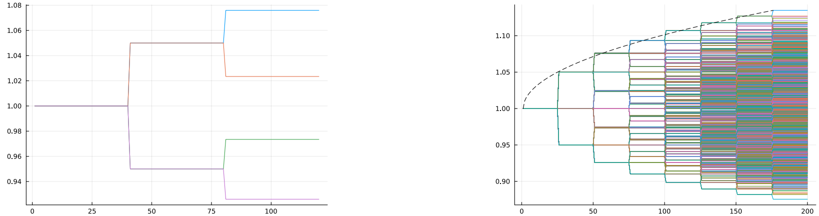
- Aggregate shocks : Complexity and Infinite dimensionality
  - The number of sequences of states/costates $(\{x_{i,t}, y_{i,t}, \mathcal{Y}_{i,t}\}, \mathcal{X}_t)$ grows with the number of trajectories, i.e. states of the world $t \rightarrow dB_t^0$

Method – Aggregate shocks

- ***Idea & Solution*** :
  Approximate & discretize dB_t^0 using a ***tree*** to follow the model/ODEs *on each trajectory*

Method – Aggregate shocks

- ***Idea & Solution*** :
  Approximate & discretize dB_t^0 using a ***tree*** to follow the model/ODEs *on each trajectory*
  - Consist of a finite set of M "waves", at dates $t_1, t_2, \dots, t_M$
  - Each wave consist of K "states of the world" ϵ for dB_t^0
  - Complexity $(B_t^0)_t$ is approximated with sequences of K^M values



Brownian motion approximated with a tree

Method – Shooting on a tree

- Stochastic case : for a "wave" $k = 1$ to M
  1. Simulate the sequence (X_t, Y_t) for $t \in [t_{k-1}, t_k]$ using the forward ODE

Method – Shooting on a tree

- Stochastic case : for a "wave" $k = 1$ to M
  1. Simulate the sequence (X_t, Y_t) for $t \in [t_{k-1}, t_k]$ using the forward ODE
  2. Simulate the sequence of costate $\tilde{Y}_{t(\epsilon)}$ for $t \in [t_k, T]$ after the realization of the shock – and for future waves – using the backward ODE.

Method – Shooting on a tree

- Stochastic case : for a "wave" $k = 1$ to M
  1. Simulate the sequence (X_t, Y_t) for $t \in [t_{k-1}, t_k]$ using the forward ODE
  2. Simulate the sequence of costate $\tilde{Y}_{t(\epsilon)}$ for $t \in [t_k, T]$ after the realization of the shock – and for future waves – using the backward ODE.
  3. Update the initial condition $Y_{t_{k-1}}$ to match the terminal condition :

$$\overline{Y}_{t_k} = \mathbb{E}\Big(\tilde{Y}_{t_k(\epsilon)} \mid \mathcal{F}_{t_{k-1}}\Big)$$

     - In practice, simulate the backward $\tilde{Y}_t$ for $t \in [t_{k-1}, t_k]$ starting from $\overline{Y}_{t_k}$ and minimize

$$\min_{Y_{t_k}} \int_{t_{k-1}}^{t_k} (\tilde{Y}_t - Y_t)^2 dt$$

     - The expectation error is expressed : $Z_{t(\epsilon)} = \mathbb{E}_\epsilon\left[\frac{\tilde{Y}_{t_k}(\epsilon) - \overline{Y}_{t_k}}{\epsilon}\right]$

Method – Shooting on a tree

- Stochastic case : for a "wave" $k = 1$ to M
  1. Simulate the sequence (X_t, Y_t) for $t \in [t_{k-1}, t_k]$ using the forward ODE
  2. Simulate the sequence of costate $\tilde{Y}_{t(\epsilon)}$ for $t \in [t_k, T]$ after the realization of the shock – and for future waves – using the backward ODE.
  3. Update the initial condition $Y_{t_{k-1}}$ to match the terminal condition :

$$\overline{Y}_{t_k} = \mathbb{E}\Big(\tilde{Y}_{t_k(\epsilon)} \mid \mathcal{F}_{t_{k-1}}\Big)$$

     – In practice, simulate the backward $\tilde{Y}_t$ for $t \in [t_{k-1}, t_k]$ starting from $\overline{Y}_{t_k}$ and minimize

$$\min_{Y_{t_k}} \int_{t_{k-1}}^{t_k} (\tilde{Y}_t - Y_t)^2 dt$$

     – The expectation error is expressed : $Z_{t(\epsilon)} = \mathbb{E}_{\epsilon}\left[\frac{\tilde{Y}_{t_k}(\epsilon) - \overline{Y}_{t_k}}{\epsilon}\right]$

  4. Redo the Forward-Backward steps 1-3 for all the waves until convergence.

The *Business as Usual* is the standard neoclassical economy

- Using PMP above, we obtain the costates More details

$$\lambda^k_{i,t} = u'(c_{i,t}) = c_{i,t}^{-\eta}$$
$$d\lambda^k_{i,t} = \lambda^k_{i,t}(\bar{\rho} - r_{i,t})dt + \mathbf{z}^k_{i,t} dB^0_t$$

The *Business as Usual* is the standard neoclassical economy

- Using PMP above, we obtain the costates More details

$$dc_{i,t} = \frac{1}{\eta} c_{i,t}(r_{i,t} - \bar{\rho})dt - c_{i,t}\mathbf{z}^c_{i,t} dB^0_t + \frac{1}{2} c_{it}(1+\eta){\mathbf{z}^c_{i,t}}^2 dt$$

The *Business as Usual* is the standard neoclassical economy

- Using PMP above, we obtain the costates More details

$$dc_{i,t} = \frac{1}{\eta} c_{i,t}(r_{i,t} - \bar{\rho})dt - c_{i,t}\mathbf{z}^c_{i,t} dB^0_t + \frac{1}{2} c_{it}(1+\eta){\mathbf{z}^c_{i,t}}^2 dt$$

- What is the impact of aggregate risk ?
  1. Direct effect : Saving/consumption on impact $\mathbf{z}_{i,t}$
  2. Indirect effect : Precautionary saving motive : $\mathbf{z}^2_{i,t}$ and prudence $1+\eta$

Impact of increase in temperature

- Using Nordhaus' Damage function $\mathcal{D}^y(\tau_{i,t}) = e^{-\frac{1}{2}\gamma_i(\tau_{i,t}-\tau_i^\star)^2}$
- Marginal values of the climate variables : $\lambda^S_{i,t}$ and $\lambda^\tau_{i,t}$

$$d\lambda^\tau_{i,t} = \Big[\lambda^\tau_{i,t}(\widetilde{\rho}+\zeta) + \overbrace{\gamma_i(\tau_{i,t}-\tau_i^\star)\mathcal{D}^y(\tau_{i,t})}^{-\partial_\tau \mathcal{D}^y(\tau_{i,t})} f(k_{i,t},e_{i,t})\lambda^k_{i,t}\Big]dt + z^\tau_{i,t}dB^0_t$$

$$d\lambda^S_{i,t} = \Big[\lambda^S_{i,t}(\widetilde{\rho}+\delta^s) - \zeta\,\chi\,\Delta_i\,\lambda^\tau_{i,t}\Big]dt + z^S_{i,t}dB^0_t$$

- Costate $\lambda^S_{i,t}$: marg. cost of $1Mt$ carbon in atmosphere, for country i. Increases with :
  - Temperature gaps $\tau_{i,t}-\tau_i^\star$ & damage sensitivity of TFP γ_i
  - Development level $f(k_{i,t},e_{i,t})$
  - Climate params : χ climate sensitivity, Δ_i “catching up” of τ_i and ζ reaction speed
  - Aggregate risk $z^\tau_{i,t}$ and $z^S_{i,t}$

Local Social cost of carbon

- The marginal "externality damage" or "local social cost of carbon" (SCC) for region i :

$$LSCC_{i,t} := -\frac{\partial \mathcal{V}_{i,t} / \partial \mathcal{S}_t}{\partial \mathcal{V}_{i,t} / \partial c_{i,t}} = -\frac{\lambda^S_{i,t}}{\lambda^k_{i,t}}$$

  - Ratio of marg. cost of carbon vs. the marg. value of consumption/capital
  - Theorem : ***Stationary LSCC*** :
    When $t \to \infty$ and for a BGP with $\mathcal{E}_t = \delta_s \mathcal{S}_t$ and $\tau_t \to \tau_\infty$, the LSCC is *proportional* to climate sensitivity χ, marg. damage γ, temperature, and output.

$$LSCC_{i,t} \equiv \frac{\chi\, \Delta_i}{\widetilde{\rho} + \delta^s} \gamma_i \left(\tau_{i,\infty} - \tau_i^\star\right) y_{i,\infty}$$

    – More general formula : Here, Proof : Here + What determine temperatures ? Details Temperature

Global Social cost of carbon

- The social planner considers a *“Global SCC”* as the marg. damage for ***all*** regions :

$$SCC_t := -\frac{\lambda_t^S}{\bar{\lambda}_t^k} = -\int_{i\in\mathbb{I}} \frac{\lambda_{i,t}^k}{\bar{\lambda}_t^k} LSCC_{i,t} di$$

  - Question : which util' unit $\bar{\lambda}_t^k$ to compute the SCC ? Average marg. utils ?

$$\bar{\lambda}_t^k = \int_{\mathbb{I}} \lambda_{j,t}^k dj$$

Global Social cost of carbon

- The social planner considers a *"Global SCC"* as the marg. damage for ***all*** regions :

$$SCC_t := -\frac{\lambda_t^S}{\bar{\lambda}_t^k} = -\int_{i\in\mathbb{I}} \frac{\lambda_{i,t}^k}{\bar{\lambda}_t^k} LSCC_{i,t} di$$

  - Question : which util' unit $\bar{\lambda}_t^k$ to compute the SCC ? Average marg. utils ?

$$\bar{\lambda}_t^k = \int_{\mathbb{I}} \lambda_{j,t}^k dj$$

  - Inequality measure :

$$\widehat{\lambda}_{i,t}^k := \frac{\lambda_{i,t}^k}{\bar{\lambda}_t^k} = \frac{\omega_i u'(c_{i,t})}{\int_{\mathbb{I}} \omega_j u'(c_{j,t}) dj} \lessgtr 1$$

Global Social cost of carbon

- The social planner considers a *"Global SCC"* as the marg. damage for ***all*** regions :

$$SCC_t := -\frac{\lambda_t^S}{\bar{\lambda}_t^k} = -\int_{i\in\mathbb{I}} \frac{\lambda_{i,t}^k}{\bar{\lambda}_t^k} LSCC_{i,t} di$$

  - Question : which util' unit $\bar{\lambda}_t^k$ to compute the SCC ? Average marg. utils ?

$$\bar{\lambda}_t^k = \int_{\mathbb{I}} \lambda_{j,t}^k dj$$

  - Inequality measure :

$$\widehat{\lambda}_{i,t}^k := \frac{\lambda_{i,t}^k}{\bar{\lambda}_t^k} = \frac{\omega_i u'(c_{i,t})}{\int_{\mathbb{I}} \omega_j u'(c_{j,t}) dj} \lesseqgtr 1$$

- This, Global SCC becomes :

$$SCC_t \equiv \mathbb{E}^{\mathbb{I}}\left[LSCC_{i,t}\right] + \mathbb{C}\mathrm{ov}^{\mathbb{I}}\left(\widehat{\lambda}_{i,t}^k, LSCC_{i,t}\right) > \mathbb{E}^{\mathbb{I}}\left[LSCC_{i,t}\right] =: \overline{SCC}_t$$

$\Rightarrow$ If damages are concentrated in high-$\widehat{\lambda}_{i,t}^k$ / poorer countries, it exacerbates the global SCC !
i.e. higher than the representative agent $SCC_t > \mathbb{E}_j[LSCC_{it}]$

Climate uncertainty and the Cost of Carbon :

- Stochastics : for any shock ϵ with distribution $\epsilon \sim \varphi(\epsilon)$
- New measure for inequalities :

$$\widehat{\widehat{\lambda}}_{it}^{k}(\epsilon) = \frac{\lambda_{it}^{k}(\epsilon)}{\mathbb{E}_{k,\epsilon}[\lambda_{i,t}^{k}(\epsilon)]} = \frac{\omega_i u'(c_{i,t}(\epsilon))}{\int_\epsilon \int_j \omega_j u'(c_{j,t}(\epsilon))\, dj\, d\varphi(\epsilon)}$$

- Uncertainty-adjusted SCC writes :

$$\mathbb{E}_\epsilon[SCC] = \int_{\mathcal{E}} \int_{\mathbb{I}} \widehat{\widehat{\lambda}}_{it}^{k}(\epsilon) LSCC_{it}(\epsilon) d\varphi(\epsilon)$$

$$= \mathbb{E}_j\Big[\underbrace{\mathbb{C}\mathrm{ov}_\epsilon\Big(\widehat{\widehat{\lambda}}_{it}^{k}(\epsilon), LSCC_{jt}(\epsilon)\Big)}_{=\text{effect of aggregate risk } \epsilon}\Big] + \underbrace{\mathbb{C}\mathrm{ov}_j\Big[\mathbb{E}_\epsilon\big(\widehat{\widehat{\lambda}}_{it}^{k}(\epsilon)\big), \mathbb{E}_\epsilon\big(LSCC_{jt}(\epsilon)\big)\Big]}_{=\text{effect of heterogeneity across } j} + \underbrace{\mathbb{E}_{j,\epsilon}[LSCC_{jt}(\epsilon)]}_{=\text{average exp. damage}}$$

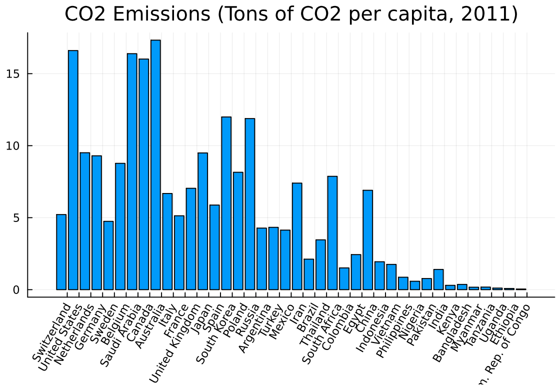
$$> \mathbb{E}_\epsilon[\overline{SCC}(\epsilon)] \qquad \& \qquad > SCC_t$$

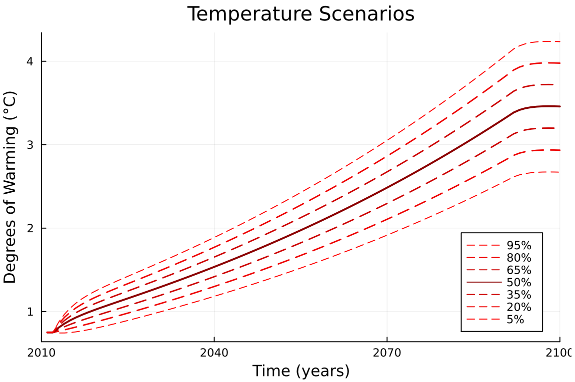
$\Rightarrow$ Climate uncertainty reinforces the unequal costs of climate change !

Numerical Application

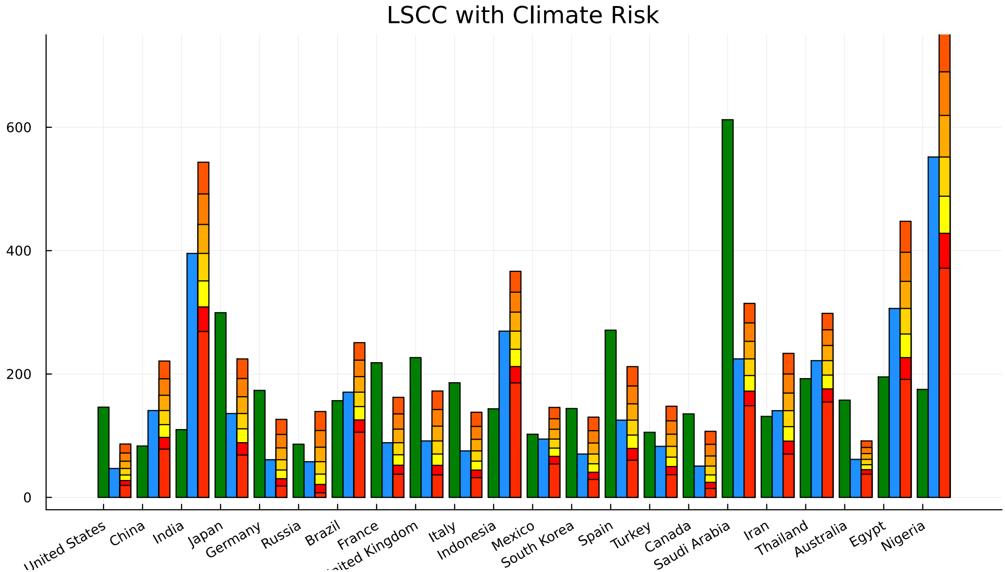
Numerical Application

- Data : 40 countries
- Temperature (of the *largest city*), GDP, energy, population
- Calibrate z to match the distribution of output per capita at steady state





Distribution of carbon prices without and with uncertainty



Conclusion

- Climate change has redistributive effects
  - Cost of carbon very heterogeneous across countries
  - Climate risk amplifies the impact on inequality
- New methodology to simulate aggregate risk globally
  - Rely on the Sequential method and shooting algorithm
  - Adapt it to aggregate risk using discretization with a tree
- Future plans :
  - More developed climate model
  - Different sources of uncertainty,
  - growth in TFP z
  - fossil/renewable price difference g^f vs g^r.

Appendices

Climate model : Extension

- Future : more sophisticated climate block – Nordhaus (2016), Cai, Lontzek, Judd (2018)
  - Emissions come from Land and Fossil

$$\mathcal{E}_t = \mathcal{E}_{\ell,t}(\mathcal{T}) + \mathcal{E}_{f,t}$$

  - World divided in "boxes" : AT : atmosphere, UO Upper Ocean+Biosphere, LO Lower Ocean

$$\mathcal{M}_t = (M_{AT,t}, M_{UO,t}, M_{LO,t}) \qquad \mathcal{T}_t = (T_{AT,t}, T_{LO,t})$$

  - Carbon Cycle, Radiative forcing and Temperature dynamics

$$d\mathcal{M}_t = \left(\Phi_M \mathcal{M}_t + (\mathcal{E}_t, 0, 0)^T\right) dt$$

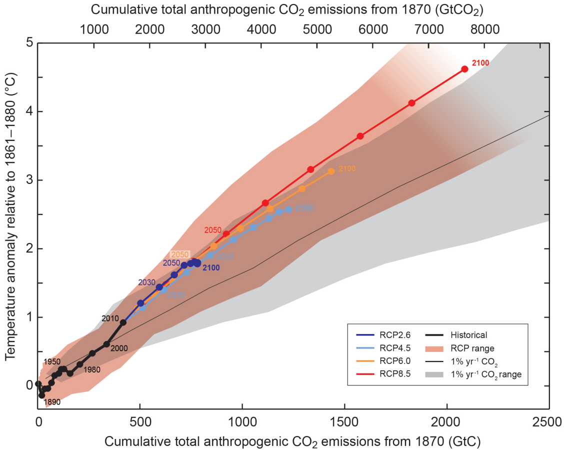
$$\mathcal{F}_t = \eta \log \left\{\frac{M_{AT,t}}{\bar{M}_{AT}}\right\} + \mathcal{F}_{ex,t}$$

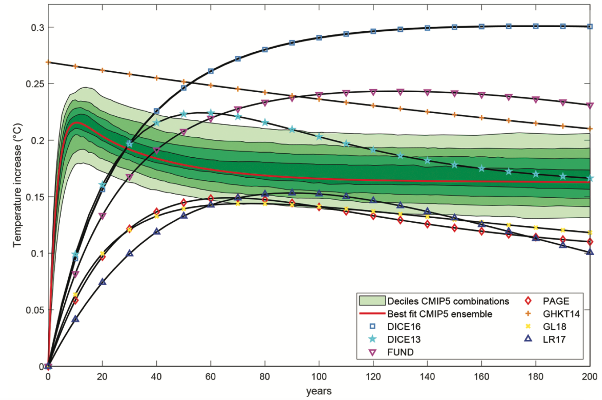
$$d\mathcal{T}_t = \left(\Phi_T \mathcal{T}_t + (\zeta \mathcal{F}_t, 0)^T\right) dt$$

  with Φ_M and Φ_T Markovian matrices

- Adding 5-6 states variables : No challenge for the sequential method at hand ! back

Temperature dynamics





Linear temperature model – IPCC report / Dietz, van der Ploeg, Rezai, Venmans (2021)

Cost of carbon / Marginal value of temperature

- Solving for the cost of carbon and temperature ⇔ solving ODE

$$\dot{\lambda}^{\tau}_{i,t} = \lambda^{\tau}_{t}(\widetilde{\rho} + \Delta\zeta) + \gamma(\tau - \tau^{\star})\mathcal{D}^{y}(\tau)f(k,e)\lambda^{k}_{t} + \phi(\tau - \tau^{\star})\mathcal{D}^{u}(\tau)u(c)$$

$$\dot{\lambda}^{S}_{t} = \lambda^{S}{}_{t}(\widetilde{\rho} + \delta^{s}) - \int_{\mathbb{I}} \Delta_i \zeta \chi \lambda^{\tau}_{i,t}$$

- Solving for λ^{τ}_{t} and λ^{S}_{t}, in stationary equilibrium $\dot{\lambda}^{S}_{t} = \dot{\lambda}^{\tau}_{t} = 0$

$$\lambda^{\tau}_{i,t} = -\int_{t}^{\infty} e^{-(\widetilde{\rho}+\zeta)u}(\tau_u - \tau^{\star})\Big(\gamma\mathcal{D}^{y}(\tau_u)y_{\tau}\lambda^{k}_{u} + \phi\mathcal{D}^{u}(\tau_u)u(c_u)\Big)du$$

$$\lambda^{\tau}_{i,t} = -\frac{1}{\widetilde{\rho} + \Delta\zeta}(\tau_{\infty} - \tau^{\star})\Big(\gamma\mathcal{D}^{y}(\tau_{\infty})y_{\infty}\lambda^{k}_{\infty} + \phi\mathcal{D}^{u}(\tau_{\infty})u(c_{\infty})\Big)$$

$$\begin{aligned}
\lambda^{S}_{t} &= -\int_{t}^{\infty} e^{-(\widetilde{\rho}+\delta^{S})u}\zeta\chi\int_{\mathbb{I}}\Delta_j\lambda^{\tau}_{j,u}dj\,du \\
&= \frac{1}{\widetilde{\rho} + \delta^{s}}\zeta\chi\int_{\mathbb{I}}\Delta_j\lambda^{\tau}_{j,\infty} \\
&= -\frac{\chi}{\widetilde{\rho} + \delta^{s}}\frac{\zeta}{\widetilde{\rho} + \zeta}\int_{\mathbb{I}}\Delta_j(\tau_{j,\infty} - \tau^{\star})\Big(\gamma\mathcal{D}^{y}(\tau_{j,\infty})y_{\infty}\lambda^{k}_{j,\infty} + \phi\mathcal{D}^{u}(\tau_{j,\infty})u(c_{j,\infty})\Big)dj
\end{aligned}$$

$$\lambda^{S}_{t} \xrightarrow[\zeta\to\infty]{} -\frac{\chi}{\widetilde{\rho} + \delta^{s}}\int_{\mathbb{I}}\Delta_j(\tau_{j,\infty} - \tau^{\star})\Big(\gamma\mathcal{D}^{y}(\tau_{j,\infty})y_{j,\infty}\lambda^{k}_{j,\infty} + \mathcal{D}^{u}(\tau_{j,\infty})u(c_{j,\infty})\Big)dj$$

Cost of carbon / Marginal value of temperature

- Closed form solution for CC :
  - In stationary equilibrium : $\dot{\lambda}_t^S = \dot{\lambda}_t^T = 0$
  - Fast temperature adjustment $\zeta \to \infty$
  - no internalization of externality (business as usual)

$$LSCC_{i,t} \equiv \frac{\Delta_i \chi}{\rho - n + \bar{g}(\eta - 1) + \delta^s}(\tau_\infty - \tau^\star)\left(\gamma \mathcal{D}^y(\tau_\infty) y_\infty + \phi \mathcal{D}^u(\tau_\infty) \frac{c_\infty}{1-\eta}\right)$$

- Heterogeneity + uncertainty about models Back

Social cost of carbon & temperature

- Cost of carbon depends only on final temperatures and path of emissions :

$$\tau_T - \tau_{t_0} = \Delta\,\chi\,\xi\,\omega \int_{t_0}^{T} e^{(n+\bar{g})t-\delta_s(T-t)} q_t^{f\ -\sigma_e} \int_{j\in\mathbb{I}} \left(z_j z_{j,t}^e \mathcal{D}(\tau_{j,t})\right)^{\sigma-1} y_{j,t}\, q_{j,t}^{\sigma_e-\sigma} dj\, dt$$

  - Geographical factors determining warming Δ_i
  - Climate sensitivity χ & carbon exit from atmosphere δ_s
  - Growth of population n, aggregate productivity $\bar{g}$
  - Deviation of output from trend y_i & relative TFP z_j
  - Directed technical change z_t^e, elasticity of energy in output σ
  - Fossil energy price $q^{e,f}$ and Hotelling rent $g^{q^f} \approx \dot{\lambda}_t^R / \lambda_t^R = \rho$
  - Change in energy mix, renewable share ω, price q_t^r & elasticity of source σ_e

- Approximations at $T \equiv$ Generalized Kaya (or $I = PAT$) identity More details

$$\frac{\dot{\tau}_T}{\tau_T} \propto\ n + \bar{g}^y - (1-\sigma)\left(g^{z^e} - \widetilde{\gamma}\right) + (\sigma_e - \sigma)(1-\omega)g^{q^r} - (\sigma_e(1-\omega) + \sigma\omega)g^{q^f}$$

Back

FBSDE for MFG systems – general formulation

- States variables :
  - Individual : $x_{i,t} \in \mathbb{X} \subset \mathbb{R}^d$ (possibly with state-constraints), with distribution $P_{x,t}$
  - Aggregate : $\mathcal{X}_t \in \overline{\mathbb{X}} \subset \mathbb{R}^d$, and controls $c^\star(\cdot) \in \mathbb{C}$

$$
\begin{aligned}
dx_{i,t} &= b(x_{i,t}, \mathcal{X}_t, c^\star_{i,t})dt + \sigma(x_{i,t}, \mathcal{X}_t)dB^0_t \\
d\mathcal{X}_t &= \bar{b}(\mathcal{X}_t, P_{x,t})dt + \bar{\sigma}(\mathcal{X}_t, P_{x,t})dB^0_t
\end{aligned}
$$

FBSDE for MFG systems – general formulation

- States variables :
  - Individual : $x_{i,t} \in \mathbb{X} \subset \mathbb{R}^d$ (possibly with state-constraints), with distribution $P_{x,t}$
  - Aggregate : $\mathcal{X}_t \in \overline{\mathbb{X}} \subset \mathbb{R}^d$, and controls $c^\star(\cdot) \in \mathbb{C}$

$$dx_{i,t} = b(x_{i,t}, \mathcal{X}_t, c^\star_{i,t})dt + \sigma(x_{i,t}, \mathcal{X}_t)dB^0_t$$
$$d\mathcal{X}_t = \bar{b}(\mathcal{X}_t, P_{x,t})dt + \bar{\sigma}(\mathcal{X}_t, P_{x,t})dB^0_t$$

- Hamiltonian :

$$\mathcal{H}(x, y, z, \mathcal{X}, \mathcal{Y}, \mathcal{Z}) = \max_{c \in \mathbb{C}} \big(u(x, c) + b(x, \mathcal{X}, c) \cdot y + \sigma(x, \mathcal{X}) * z\big) + \bar{b}(\mathcal{X}_t, P_{x,t}) \cdot \mathcal{Y} + \bar{\sigma}(\mathcal{X}_t, P_{x,t}) * \mathcal{Z}$$

- Optimal control $c^\star \in \text{argmax}_{c \in \mathbb{C}} \big(u(x, c) + b(x, \mathcal{X}, c) \cdot y\big)$

FBSDE for MFG systems – general formulation

- States variables :
  - Individual : $x_{i,t} \in \mathbb{X} \subset \mathbb{R}^d$ (possibly with state-constraints), with distribution $P_{x,t}$
  - Aggregate : $\mathcal{X}_t \in \overline{\mathbb{X}} \subset \mathbb{R}^d$, and controls $c^\star(\cdot) \in \mathbb{C}$

$$dx_{i,t} = b(x_{i,t}, \mathcal{X}_t, c^\star_{i,t})dt + \sigma(x_{i,t}, \mathcal{X}_t)dB^0_t$$
$$d\mathcal{X}_t = \bar{b}(\mathcal{X}_t, P_{x,t})dt + \bar{\sigma}(\mathcal{X}_t, P_{x,t})dB^0_t$$

- Hamiltonian :

$$\mathcal{H}(x, y, z, \mathcal{X}, \mathcal{Y}, \mathcal{Z}) = \max_{c\in\mathbb{C}} \big(u(x, c) + b(x, \mathcal{X}, c) \cdot y + \sigma(x, \mathcal{X}) * z\big) + \bar{b}(\mathcal{X}_t, P_{x,t}) \cdot \mathcal{Y} + \bar{\sigma}(\mathcal{X}_t, P_{x,t}) * \mathcal{Z}$$

- Optimal control $c^\star \in \text{argmax}_{c\in\mathbb{C}} \big(u(x, c) + b(x, \mathcal{X}, c) \cdot y\big)$
- Using the Stochastic PMP :

$$dy_{i,t} = -D_x\mathcal{H}(x_{i,t}, y_{i,t}, z_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}, \mathcal{Z}_{i,t})dt + z_{i,t}dB^0_t$$
$$d\mathcal{Y}_{i,t} = -D_X\mathcal{H}(x_{i,t}, y_{i,t}, z_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}, \mathcal{Z}_{i,t})dt + \mathcal{Z}_{i,t}dB^0_t$$

Model solution : FBSDE system for MFG

- Coupled FBSDE system for each agent

$$\begin{cases} dx_{i,t} &= D_y \mathcal{H}(x_{i,t}, y_{i,t}, z_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}, \mathcal{Z}_{i,t}) dt + \sigma(x_{i,t}, \mathcal{X}_t) dB_t^0 \\ dy_{i,t} &= -D_x \mathcal{H}(x_{i,t}, y_{i,t}, z_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}, \mathcal{Z}_{i,t}) dt + z_{i,t} dB_t^0 \end{cases}$$

- Question : What else do we need ?

Model solution : FBSDE system for MFG

- Coupled FBSDE system for each agent

$$\begin{cases} dx_{i,t} &= D_y \mathcal{H}(x_{i,t}, y_{i,t}, z_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}, \mathcal{Z}_{i,t})dt + \sigma(x_{i,t}, \mathcal{X}_t)dB_t^0 \\ dy_{i,t} &= -D_x \mathcal{H}(x_{i,t}, y_{i,t}, z_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}, \mathcal{Z}_{i,t})dt + z_{i,t} dB_t^0 \end{cases}$$

- Question : What else do we need ?
  - The individual risk loading in the costate $z_{i,t}$:
    - Expectation error in the law of motion of $y_{i,t}$

$$z_{i,t}(x, \mathcal{X}, y) = \mathbb{E}^{\epsilon}\left[\frac{y_{i,t+dt}(\epsilon) - y_{i,t} + D_x \mathcal{H}(x_{i,t}, y_{i,t}, z_{i,t}, \mathcal{X}_t)dt}{dB_t^0}\right]$$

    - BSDE theory : keep the co-state measurable w.r.t. dB_t^0, despite running backward.
      $\Rightarrow$ ***Intuition :*** even if agents are forward-looking, they can't know the future.
    - Advantage : Numerically Feasible via Monte Carlo or Tree Methods

Model solution : FBSDE system for MFG

- Coupled FBSDE system for each agent

$$\begin{cases} dx_{i,t} & = D_y \mathcal{H}(x_{i,t}, y_{i,t}, z_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}, \mathcal{Z}_{i,t}) dt + \sigma(x_{i,t}, \mathcal{X}_t) dB_t^0 \\ dy_{i,t} & = -D_x \mathcal{H}(x_{i,t}, y_{i,t}, z_{i,t}, \mathcal{X}_t, \mathcal{Y}_{i,t}, \mathcal{Z}_{i,t}) dt + z_{i,t} dB_t^0 \end{cases}$$

- Question : What else do we need ?
  - The individual risk loading in the costate $z_{i,t}$:
    - Expectation error in the law of motion of $y_{i,t}$

$$z_{i,t}(x, \mathcal{X}, y) = \mathbb{E}^{\epsilon}\left[\frac{y_{i,t+dt}(\epsilon) - y_{i,t} + D_x \mathcal{H}(x_{i,t}, y_{i,t}, z_{i,t}, \mathcal{X}_t) dt}{dB_t^0}\right]$$

    - BSDE theory : keep the co-state measurable w.r.t. dB_t^0, despite running backward.
      $\Rightarrow$ ***Intuition :*** even if agents are forward-looking, they can't know the future.
    - Advantage : Numerically Feasible via Monte Carlo or Tree Methods
  - The initial condition y_0 as a function of y_0

Model solution : FBSDE system for MFG

- Coupled FBSDE system for each agent

$$\begin{cases} dx_{i,t} & = D_y\mathcal{H}(x_{i,t},y_{i,t},z_{i,t},\mathcal{X}_t,\mathcal{Y}_{i,t},\mathcal{Z}_{i,t})dt + \sigma(x_{i,t},\mathcal{X}_t)dB_t^0 \\ dy_{i,t} & = -D_x\mathcal{H}(x_{i,t},y_{i,t},z_{i,t},\mathcal{X}_t,\mathcal{Y}_{i,t},\mathcal{Z}_{i,t})dt + z_{i,t}dB_t^0 \end{cases}$$

- Question : What else do we need ?
  - The individual risk loading in the costate $z_{i,t}$:
    - Expectation error in the law of motion of $y_{i,t}$

$$z_{i,t}(x,\mathcal{X},y) = \mathbb{E}^{\epsilon}\left[\frac{y_{i,t+dt}(\epsilon) - y_{i,t} + D_x\mathcal{H}(x_{i,t},y_{i,t},z_{i,t},\mathcal{X}_t)dt}{dB_t^0}\right]$$

    - BSDE theory : keep the co-state measurable w.r.t. dB_t^0, despite running backward.
      $\Rightarrow$ ***Intuition :*** even if agents are forward-looking, they can't know the future.
    - Advantage : Numerically Feasible via Monte Carlo or Tree Methods
  - The initial condition y_0 as a function of y_0
  - A boundary condition of y_T or transversality $\lim_{t\to\infty} e^{-\rho t}x_t y_t = 0$

back

FBSDE for McKean Vlasov systems – general formulation

- Let us consider the Social Planner :

$$\mathcal{W}_{t_0} = \max_{\{c_i\}_i} \int_{t_0}^{\infty} e^{-\rho t} \int_{i\in\mathbb{I}} \omega_i \, u_i(x_{i,t}, c_{i,t}) di \, dt$$

s.t. individual *and* aggregate dynamics, and controlling $c_i, \forall i \in \mathbb{I}$.

FBSDE for McKean Vlasov systems – general formulation

- Let us consider the Social Planner :

$$\mathcal{W}_{t_0} = \max_{\{c_i\}_i} \int_{t_0}^{\infty} e^{-\rho t} \int_{i \in \mathbb{I}} \omega_i \, u_i(x_{i,t}, c_{i,t}) di \, dt$$

s.t. individual *and* aggregate dynamics, and controlling $c_i, \forall i \in \mathbb{I}$.

- Set up the Social Planner Hamiltonian :

$$\bar{\mathcal{H}}^{SP}(\{x, y, z\}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}, P_x) = \max_{c \in \mathbb{C}} \int_{x \in \mathbb{X}} \Big[\omega u(x, c) + b(x, \mathcal{X}, c) \cdot y + \sigma(x, \mathcal{X}) * z \Big] P_x(dx) + \bar{b}(\mathcal{X}, P_x) \cdot \mathcal{Y} + \bar{\sigma}(\mathcal{X}, P_x) * \mathcal{Z}$$

- Optimal control $c^{\star} \in \text{argmax}_{x \in \mathbb{C}} \bar{\mathcal{H}}(\cdot)$

FBSDE for McKean Vlasov systems – general formulation

- Using the Stochastic Pontryagin maximum principle :

$$dy_{i,t} = -D_x\bar{\mathcal{H}}^{SP}(\{x,y,z\},\mathcal{X},\mathcal{Y},\mathcal{Z},P_x)dt + \tilde{z}_{i,t}dB_t^0 - \widetilde{\mathbb{E}}\left[D_\mu\bar{\mathcal{H}}^{SP}(\{\tilde{x},\tilde{y},\tilde{z}\},\mathcal{X},\mathcal{Y},\mathcal{Z},P_x)(x_{i,t})\right]dt$$

$$d\mathcal{Y}_t = -D_X\bar{\mathcal{H}}^{SP}(\{x,y,z\},\mathcal{X},\mathcal{Y},\mathcal{Z},P_x)dt + \mathcal{Z}_t dB_t^0$$

Two effects internalized by the social planner :

1. Effect on Aggregate variables $\mathcal{X}_t$
2. Effect on the distribution P_x :

FBSDE for McKean Vlasov systems – general formulation

- Using the Stochastic Pontryagin maximum principle :

$$dy_{i,t} = -D_x\bar{\mathcal{H}}^{SP}(\{x,y,z\},\mathcal{X},\mathcal{Y},\mathcal{Z},P_x)dt + \tilde{z}_{i,t}dB_t^0 - \widetilde{\mathbb{E}}\left[D_\mu\bar{\mathcal{H}}^{SP}(\{\tilde{x},\tilde{y},\tilde{z}\},\mathcal{X},\mathcal{Y},\mathcal{Z},P_x)(x_{i,t})\right]dt$$
$$d\mathcal{Y}_t = -D_X\bar{\mathcal{H}}^{SP}(\{x,y,z\},\mathcal{X},\mathcal{Y},\mathcal{Z},P_x)dt + \mathcal{Z}_t dB_t^0$$

Two effects internalized by the social planner :

1. Effect on Aggregate variables $\mathcal{X}_t$
2. Effect on the distribution P_x :
    - Intuition : shifting the distribution of states x for all other agents $\tilde{x}$
    - $D_\mu H$ is the L-derivative w.r.t the measure $\mu \equiv P_{x,t}$
    - Idea : lifting of the function $H(x,\mu) = \widehat{H}(x,\widehat{X})$ where $\widehat{X} \sim \mu$ and hence $D_\mu H(x,\mu)(\widehat{X}) = D_{\widehat{x}}\widetilde{H}(x,\widehat{X})$
    - Probabilistic approach : easy to compute $\widetilde{\mathbb{E}}[D_\mu H(\tilde{x}_t,\mu)] = \widetilde{\mathbb{E}}[D_{\widehat{x}}H(\tilde{x}_t,\widehat{X})]$
    - Here : effects are homogeneous for all agents : interaction with measure P_x is non-local !
    - Back

More details – PMP – Competitive equilibrium

- Household problem : State variables $x_{i,t} = (k_i, \tau_i)$ Back summary, Back explanation
- Pontryagin Maximum Principle

$$\mathcal{H}(x, \{c, e^f, e^r\}, \{\lambda^k, \lambda^\tau, \lambda^s\}) = u(c_i, \tau_i) + \lambda^k_{i,t}\Big(\mathcal{D}(\tau_{it})f(k_t, e_t) - (n + \bar{g} + \delta)k_t - q^f_t e^f_{it} - q^r_{it} e^r_{it} - c_t\Big)$$
$$+ \lambda^\tau_{i,t}\zeta\Big(\Delta_i \chi \mathcal{S}_t - (\tau_{it} - \tau_{i0})\Big) + \lambda^S_{i,t}\Big(\mathcal{E}_t - \delta^s \mathcal{S}_t\Big)$$

More details – PMP – Competitive equilibrium

- Household problem : State variables $x_{i,t} = (k_i, \tau_i)$ Back summary, Back explanation
- Pontryagin Maximum Principle

$$\mathcal{H}(x, \{c, e^f, e^r\}, \{\lambda^k, \lambda^\tau, \lambda^s\}) = u(c_i, \tau_i) + \lambda^k_{i,t}\Big(\mathcal{D}(\tau_{it})f(k_t, e_t) - (n + \bar{g} + \delta)k_t - q^f_t e^f_{it} - q^r_{it} e^r_{it} - c_t\Big)$$
$$+ \lambda^\tau_{i,t}\zeta\Big(\Delta_i\, \chi\, \mathcal{S}_t - (\tau_{it} - \tau_{i0})\Big) + \lambda^S_{i,t}\Big(\mathcal{E}_t - \delta^s \mathcal{S}_t\Big)$$

$$u'(c_{it}) = \lambda^k_{i,t}$$

$$[e^f_t] \qquad MPe^f_{it} = \mathcal{D}(\tau_{i,t}) z\, \partial_e f(k_{i,t}, e_{i,t})\left(\frac{e^f_{i,t}}{\omega e_{i,t}}\right)^{-\frac{1}{\sigma_e}} = q^f_t$$

$$[e^r_t] \qquad MPe^r_{it} = \mathcal{D}(\tau_{i,t}) z\, \partial_e f(k_{i,t}, e_{i,t})\left(\frac{e^r_{i,t}}{(1-\omega) e_{i,t}}\right)^{-\frac{1}{\sigma_e}} = q^r_{it}$$

$$[k_{i,t}] \qquad \dot{\lambda}^k_t = \lambda^k_t\big(\rho - \partial_k f(k_{i,t}, e_{i,t})\big)$$

$$[\tau_{i,t}] \qquad \dot{\lambda}^\tau_{i,t} = \lambda^\tau_{i,t}(\widetilde{\rho} + \zeta) + \overbrace{\gamma_i(\tau_{i,t} - \tau^\star_i)\mathcal{D}^y(\tau_{i,t})}^{-\partial_\tau \mathcal{D}^y} f(k_{i,t}, e_{i,t})\lambda^k_{i,t} + \overbrace{\phi_i(\tau_{i,t} - \tau^\star_i)\mathcal{D}^u(\tau_{i,t})}^{\partial_\tau \mathcal{D}^u} u(c_{i,t})$$

$$[\mathcal{S}_t] \qquad \dot{\lambda}^S_{i,t} = \lambda^S_{i,t}(\widetilde{\rho} + \delta^s) - \zeta\, \chi\, \Delta_i\, \lambda^\tau_{i,t}$$

More details – PMP – Ramsey Optimal Allocation

- Hamiltonian :

$$
\begin{aligned}
\mathcal{H}^{sp}(s,\{c\},\{e^f\},\{e^r\},\{\lambda\},\{\psi\}) = \int_{\mathbb{I}} \omega_i \mathcal{D}^u(\tau_{it}) u(c_i) p_i di + \\
+ \psi^k_{i,t}\Big(\mathcal{D}(\tau_{it}) f(k_t, e_t) - (n+\bar{g}+\delta)k_t + \theta_i \pi(E^f_t, \mathcal{I}_t, \mathcal{R}_t) - q^f_t e^f_{it} - q^r_{it} e^r_{it} - c_t\Big) \\
+ \psi^S_t\Big(\mathcal{E}_t - \delta^s \mathcal{S}_t\Big) + \psi^\tau_{it}\zeta\Big(\Delta_i \chi \mathcal{S}_t - (\tau_{it} - \tau_{i0})\Big) + \psi^{\mathcal{R}}_{it}\Big(-E^f_t + \delta^R \mathcal{I}_t\Big) \\
+ \psi^{\lambda k}_{i,t}\Big(\lambda^k_t(\rho - r_t)\Big) + \psi^{\lambda R}_t\Big(\rho\lambda^R_t + \mathcal{C}^f_{\mathcal{R}}(E^f_t, \mathcal{I}_t, \mathcal{R}_t)\Big) \\
+ \phi^c_{it}\big(\mathcal{D}^u(\tau_{it}) u'(c_i) - \lambda^k_{it}\big) + \phi^{ef}_{it}\Big(MPe^f_{it} - q^f_t\Big) + \phi^r_{it}\Big(MPe^r_{it} - q^r_{it}\Big) \\
+ \phi^{Ef}_t\Big(q^f_t - \mathcal{C}^f_E(\cdot) - \lambda^{\mathcal{R}}_t\Big) + \phi^{\mathcal{I}f}_t\big(\delta\lambda^{\mathcal{R}}_t - \mathcal{C}^f_{\mathcal{I}}(\cdot)\big)
\end{aligned}
$$

- Back

Ramsey Optimal Allocation - FOCs

- FOCs

$$[c_{it}] \qquad \psi^k_{it} = \underbrace{\omega_i \mathcal{D}^u(\tau_{it}) u'(c_i) p_i}_{=\text{direct effect}} + \underbrace{\phi^c_{it} \mathcal{D}^u(\tau_{it}) u''(c_i)}_{=\text{effect on savings}}$$

$$\text{Define :} \qquad \widehat{\phi}^e_{it} = \phi^f_{it} MPe^f_t + \phi^r_{it} MPe^r_t$$

$$[e^f_{it}] \qquad \psi^k_{i,t}\left(MPe^f_{it} - q^f_t\right) + \xi_{it} p_i \psi^S_t + p_i \partial_E \pi^f(\cdot) \int_{\mathbb{I}} \theta_j \psi^k_{jt} dj + \partial_{e^f} \widehat{\phi}^e_{it} - p_i \phi^{Ef}_t \partial_{EE} \mathcal{C}(\cdot) = 0$$

$$[e^r_{it}] \qquad \psi^k_{i,t}\left(MPe^r_{it} - q^r_{it}\right) + \partial_{e^r} \widehat{\phi}^e_{it} = 0$$

$$[\mathcal{I}_t] \qquad \delta\, \psi^{\mathcal{R}}_t + \partial^2_{\mathcal{R}\mathcal{I}} \mathcal{C}(\cdot)\, \psi^{\lambda,\mathcal{R}}_t - \phi^{\mathcal{I}}_t \partial^2_{\mathcal{I}\mathcal{I}} \mathcal{C}(\cdot) = 0$$

$$[q^f_t] \qquad \phi^{Ef}_t = \int_{\mathbb{I}} e^f_{it} \psi^k_{jt} dj + \int_{\mathbb{I}} \phi^f_{jt} dj - \partial_{q^f} \pi^f(\cdot) \int_{\mathbb{I}} \theta_j \psi^k_{jt} dj$$

- Back

Ramsey Optimal Allocation - FOCs

- Backward equations for planner's costates

$$[k_i] \qquad \dot{\psi}^k_{it} = \psi^k_{it}(\tilde{\rho} - r_{it} + \partial_k MPk_i)\psi^k_{it} - \partial_k \widehat{\phi}^e_{it}$$

$$[\mathcal{S}_i] \qquad \dot{\psi}^{\mathcal{S}}_t = (\tilde{\rho} + \delta^s)\psi^{\mathcal{S}}_t - \int_{\mathbb{I}} \Delta_j \zeta \chi \psi^{\tau}_{jt} dj$$

$$[\tau_i] \qquad \dot{\psi}^{\tau}_t = (\tilde{\rho} + \zeta)\psi^{\tau}_t - \Big(\omega_i \mathcal{D}'(\tau_{it})u(c_{it}) + \psi^k_{it}\mathcal{D}'(\tau_{it})f(k_{it}, e_{it}) + \phi^c_{it}\mathcal{D}'(\tau_{it})u'(c_i) + \partial_\tau \widehat{\phi}^e_{it}\Big)$$

$$[\mathcal{R}] \qquad \dot{\psi}^{\mathcal{R}}_t = \psi^{\mathcal{R}}_t\Big(\tilde{\rho} - \partial^2_{\mathcal{R}\mathcal{R}}\mathcal{C}(\cdot)\Big) - \phi^{Ef}_t \partial^2_{\mathcal{R}E}\mathcal{C}(\cdot)$$

$$[\lambda^k_i] \qquad \dot{\psi}^{\lambda,k}_t = \tilde{\rho}\psi^{\lambda,k}_t - (\rho - r_{i,t})\psi^k_t + \phi^c_{i,t}$$

$$[\lambda^{\mathcal{R}}_i] \qquad \dot{\psi}^{\lambda,\mathcal{R}}_t = \psi^{\lambda,\mathcal{R}}_t(\tilde{\rho} - \rho) + \phi^{Ef}_t - \delta\phi^{\mathcal{I}f}_t$$

- Back