Advanced Macroeconomics – Part 2 – Theme 1 (Lectures 1–2)

Problem set: Business Cycle Accounting and the New Keynesian Model – Answer key

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Scope. This problem set covers Theme 1 and has two parts.

- **Part I – Business cycle accounting** (Exercises 1–3). Each exercise takes an economy with a specific friction and shows that its aggregate allocations are those of a prototype growth model with time-varying wedges.
- **Part II – The New Keynesian model** (Exercises 4–7). The exercises cover a two-period model with a fixed price, the derivation of the Phillips curve under Calvo pricing, the welfare effects of demand under monopolistic competition, and the time inconsistency of monetary policy.

The parts are linked in two places: the markup shock of Exercise 3 is the cost-push shock of Part II, and Exercises 4 and 6 use the wedges of Part I.

How to use it. The sheet is long and is not meant to be done in one sitting. Each exercise can be done on its own; when one uses a result from another, the reference is given. Each part starts with a reminder of its notation.

Contents.

1. Input-financing frictions → efficiency wedge — *CKM (2007), Prop. 1*
2. Sticky nominal wages → labor wedge — *Bordo, Erceg & Evans (2000); CKM, Prop. 2*
3. Markup shocks → labor and investment wedges
4. Sticky prices in a two-period model — *Mankiw & Weinzierl (2011)*
5. From the Calvo system to the New Keynesian Phillips curve — *Calvo (1983); Galí (2015), ch. 3*
6. Monopolistic competition and the aggregate demand externality — *Blanchard & Kiyotaki (1987)*
7. Time inconsistency and the inflation bias — *Kydland & Prescott (1977)*

Part I – Business cycle accounting

Objective. Chari, Kehoe & McGrattan (2007, *Econometrica*, henceforth CKM) show in their Section 1.2 that economies with specific frictions are *equivalent* to the prototype growth model with time-varying wedges. The three exercises of this part derive such results step by step. Input-financing frictions show up as an *efficiency wedge* (Exercise 1), sticky nominal wages as a *labor wedge* (Exercise 2), and markup shocks as *labor and investment wedges* (Exercise 3).

Reminder: the prototype economy. Abstracting from growth, the equilibrium of the prototype economy is summarized by

$$c_t + x_t + g_t = y_t, \qquad y_t = A_t F(k_t, l_t),$$
$$-\frac{U_{lt}}{U_{ct}} = (1-\tau_{lt})\, A_t F_{lt}, \qquad U_{ct}(1+\tau_{xt}) = \beta\, \mathbb{E}_t\Big[U_{ct+1}\big\{A_{t+1}F_{kt+1} + (1-\delta)(1+\tau_{xt+1})\big\}\Big],$$

with efficiency wedge A_t, labor wedge $1-\tau_{lt}$, investment wedge $1/(1+\tau_{xt})$ and government consumption wedge g_t. Distortionary "tax" revenues are rebated lump sum. *Equivalence* means: for a given detailed economy, we can choose wedge processes such that the aggregate allocations of the detailed economy are an equilibrium of the prototype economy.

Exercise 1 – Input-financing frictions → efficiency wedge

The detailed economy. Time is discrete and there is no aggregate uncertainty within the period. There are two sectors $i \in \{1, 2\}$.

- A competitive final-good producer combines sectoral outputs into gross output $q_t = q_{1t}^{\phi}\, q_{2t}^{1-\phi}$, $\phi \in (0,1)$. The price of q_t is normalized to 1; p_{it} is the price of sector-i output.
- Sector i produces $q_{it} = m_{it}^{\theta}\, z_{it}^{1-\theta}$, $\theta \in (0,1)$, using intermediate goods m_{it} (one unit of q_t each) and a composite value-added good z_{it} with price v_t. Intermediate goods must be *financed in advance* within the period, at gross rate $R_{it} = 1+\tau_{it}$, where $\tau_{it} \geq 0$ is a spread induced by financing constraints (the within-period rate paid to savers is 1). Sector-i producers solve

$$\max_{m_{it}, z_{it}} \; p_{it} q_{it} - v_t z_{it} - (1+\tau_{it})\, m_{it}.$$

- A competitive producer makes the composite good from capital and labor, $z_{1t} + z_{2t} = z_t = F(k_t, l_t)$, with F constant returns to scale, and solves max $v_t z_t - w_t l_t - r_t k_t$.
- Consumers maximize $\sum_t \beta^t U(c_t, l_t)$ subject to $c_t + k_{t+1} = r_t k_t + w_t l_t + (1-\delta)k_t + T_t$, where $T_t = \sum_i \tau_{it} m_{it}$ rebates the spreads lump sum.
- Resource constraint: $c_t + k_{t+1} + m_{1t} + m_{2t} = q_t + (1-\delta)k_t$. Final output (value added) is $y_t = q_t - m_{1t} - m_{2t}$.

Throughout, define $a_{1t} = \dfrac{\phi}{1+\tau_{1t}}$ and $a_{2t} = \dfrac{1-\phi}{1+\tau_{2t}}$.

Part A – Solving the detailed economy

1. Final-good aggregator. Solve the problem of the final-good producer, max $q_t - p_{1t}q_{1t} - p_{2t}q_{2t}$, and show that expenditure shares are constant: $p_{1t}q_{1t} = \phi\, q_t$ and $p_{2t}q_{2t} = (1-\phi)\, q_t$.

Answer. FOCs: $\phi q_t/q_{1t} = p_{1t}$ and $(1-\phi)q_t/q_{2t} = p_{2t}$. Multiplying by q_{it} gives the result (standard Cobb-Douglas expenditure shares).

2. Sectoral demands. Write down the first-order conditions of the sector-i producer. Using Question 1, show that

$$m_{it} = \theta\, a_{it}\, q_t, \qquad z_{1t} = \phi\, z_t, \qquad z_{2t} = (1-\phi)\, z_t, \qquad v_t = (1-\theta)\frac{q_t}{z_t}.$$

Answer. FOCs: $(1+\tau_{it})m_{it} = \theta\, p_{it}q_{it}$ and $v_t z_{it} = (1-\theta)\, p_{it}q_{it}$. With $p_{1t}q_{1t} = \phi q_t$: $m_{1t} = \theta\phi q_t/(1+\tau_{1t}) = \theta a_{1t}q_t$, and similarly for sector 2. Summing the second FOC over i: $v_t z_t = (1-\theta)q_t$, so $v_t = (1-\theta)q_t/z_t$, and $z_{it}/z_t = p_{it}q_{it}/q_t$, i.e. ϕ and $1-\phi$. Note that the spreads distort the use of intermediates, but *not* the allocation of value added across sectors.

3. Final output. Show that final output is

$$y_t = q_t\big[1 - \theta(a_{1t} + a_{2t})\big].$$

What is y_t/q_t without frictions ($\tau_{1t} = \tau_{2t} = 0$)? Why do frictions *raise* y_t/q_t?

Answer. $y_t = q_t - m_{1t} - m_{2t} = q_t - \theta(a_{1t}+a_{2t})q_t$. Without frictions $a_{1t} + a_{2t} = 1$, so $y_t/q_t = 1-\theta$. Positive spreads make intermediates more expensive, so firms use fewer of them per unit of gross output and the value-added share of gross output rises. Of course, q_t itself falls (see Question 4).

4. Aggregate production function. Substitute the demands of Question 2 into $q_t = q_{1t}^{\phi} q_{2t}^{1-\phi}$ and solve for q_t as a function of z_t. Deduce that

$$y_t = A_t\, F(k_t, l_t), \qquad A_t = \kappa\left(a_{1t}^{\phi}\, a_{2t}^{1-\phi}\right)^{\frac{\theta}{1-\theta}}\big[1 - \theta(a_{1t}+a_{2t})\big], \qquad \kappa = \theta^{\frac{\theta}{1-\theta}}\phi^{\phi}(1-\phi)^{1-\phi}.$$

Answer. $q_{1t} = (\theta a_{1t} q_t)^{\theta}(\phi z_t)^{1-\theta}$ and $q_{2t} = (\theta a_{2t} q_t)^{\theta}((1-\phi)z_t)^{1-\theta}$. Hence

$$q_t = \theta^{\theta} q_t^{\theta}\left(a_{1t}^{\phi} a_{2t}^{1-\phi}\right)^{\theta}\left(\phi^{\phi}(1-\phi)^{1-\phi}\right)^{1-\theta} z_t^{1-\theta} \quad \Rightarrow \quad q_t = \theta^{\frac{\theta}{1-\theta}}\phi^{\phi}(1-\phi)^{1-\phi}\left(a_{1t}^{\phi} a_{2t}^{1-\phi}\right)^{\frac{\theta}{1-\theta}} z_t.$$

Multiply by $1-\theta(a_{1t}+a_{2t})$ and use $z_t = F(k_t, l_t)$. Even though both sectoral technologies are constant, measured TFP A_t depends on the spreads. *Remark:* the expression printed in CKM (2007), eq. (12), has the exponents of a_{1t} and a_{2t} swapped and a different constant; the version above is the one implied by the model (the constant is irrelevant for the argument).

5. Factor prices and the labor wedge. Using the composite producer's problem and Question 2, show that

$$w_t = v_t F_{lt}, \qquad r_t = v_t F_{kt}, \qquad v_t = \frac{1-\theta}{1-\theta(a_{1t}+a_{2t})}\, A_t.$$

Consider a prototype economy in which consumers face taxes on labor income τ_{lt} and on capital income τ_{kt}, and firms pay factors their marginal products $A_t F_{lt}$ and $A_t F_{kt}$. Show that the consumers' first-order conditions coincide in the two economies if and only if

$$\frac{1}{1-\tau_{lt}} = \frac{1}{1-\theta}\big[1 - \theta(a_{1t}+a_{2t})\big], \qquad \tau_{kt} = \tau_{lt}.$$

Why is the distortion the *same* for labor and capital? Show that $\tau_{lt} \geq 0$ whenever spreads are non-negative.

Answer. The composite producer's FOCs give $w_t = v_t F_{lt}$, $r_t = v_t F_{kt}$. From Questions 2–4, $v_t = (1-\theta)q_t/z_t = (1-\theta)\dfrac{y_t}{[1-\theta(a_{1t}+a_{2t})]F(k_t,l_t)} = \dfrac{1-\theta}{1-\theta(a_{1t}+a_{2t})}A_t$. In the detailed economy, the consumer's labor FOC is $-U_{lt}/U_{ct} = w_t = v_t F_{lt}$; in the prototype, $-U_{lt}/U_{ct} = (1-\tau_{lt})A_t F_{lt}$. They coincide iff $1-\tau_{lt} = v_t/A_t$, which is the stated formula. The same comparison for the Euler equation ($r_t = v_t F_{kt}$ vs. $(1-\tau_{kt})A_t F_{kt}$) gives $1-\tau_{kt} = v_t/A_t$, i.e. $\tau_{kt} = \tau_{lt}$. Both factors are paid the price of the composite good, v_t, times their marginal product in z_t, while their marginal contribution to final output y_t is A_t times that marginal product. The friction drives a wedge between v_t and A_t, which hits both factors symmetrically. If $\tau_{it} \geq 0$, then $a_{1t} + a_{2t} \leq \phi + (1-\phi) = 1$, so $1-\theta(a_{1t}+a_{2t}) \geq 1-\theta$ and $\tau_{lt} \geq 0$, with equality iff both spreads are zero.

6. **Budget and resource constraints.** Show that, in the detailed economy, consumer income $r_t k_t + w_t l_t + T_t$ equals y_t (all producers make zero profits). Show that the resource constraint can be written $c_t + k_{t+1} = y_t + (1-\delta)k_t$. Conclude that the aggregate allocations (c_t, k_{t+1}, l_t, y_t) of the detailed economy are an equilibrium of the prototype economy with A_t from Question 4, $\tau_{lt} = \tau_{kt}$ from Question 5, $g_t = 0$ and tax revenues rebated lump sum. This is CKM's **Proposition 1**.

   Answer. By constant returns, $r_t k_t + w_t l_t = v_t z_t = (1-\theta)q_t$. Summing the FOCs for intermediates, $\sum_i (1+\tau_{it}) m_{it} = \theta \sum_i p_{it} q_{it} = \theta q_t$, so $T_t = \sum_i \tau_{it} m_{it} = \theta q_t - m_{1t} - m_{2t}$. Hence income $= (1-\theta)q_t + \theta q_t - m_{1t} - m_{2t} = y_t$. The resource constraint follows from the definition of y_t. In the prototype economy, income is $(1-\tau_{kt})r_t^p k_t + (1-\tau_{lt})w_t^p l_t + T_t^p = A_t F(k_t, l_t) = y_t$ as well, with the factor prices $r_t^p = A_t F_{kt}$, $w_t^p = A_t F_{lt}$. Both economies share the same resource constraint, the same production function $y_t = A_t F(k_t, l_t)$ and, by Question 5, the same consumer FOCs: they have the same equilibrium allocations.

Part B – Relative distortions look like technological regress

Suppose now that the spreads fluctuate but the weighted average $S \equiv a_{1t} + a_{2t}$ stays constant.

7. **Which wedges move?** Using Questions 4–5, show that the labor and capital wedges are constant, while A_t moves only through $G_t \equiv a_{1t}^{\phi} a_{2t}^{1-\phi}$.

   Answer. $\tau_{lt} = \tau_{kt}$ depends on the spreads only through $a_{1t} + a_{2t} = S$, so it is constant. In A_t, the term $[1-\theta S]$ is constant, so $A_t \propto G_t^{\theta/(1-\theta)}$.

8. **Dispersion lowers measured TFP.** Holding S fixed, show that G_t is maximized when $\tau_{1t} = \tau_{2t}$. Interpret this result in terms of the allocation of intermediate goods across sectors.

   Answer. Maximize $\phi \ln a_1 + (1-\phi)\ln a_2$ subject to $a_1 + a_2 = S$: the FOC gives $\phi/a_1 = (1-\phi)/a_2$, i.e. $a_1 = \phi S$, $a_2 = (1-\phi)S$, i.e. $1+\tau_1 = 1+\tau_2 = 1/S$. With equal spreads, the ratio of marginal products of intermediates is the same in both sectors and intermediates are efficiently allocated across sectors (given their overall level). Any dispersion in spreads misallocates intermediates: the more constrained sector uses too few, the other too many, and output falls with no change in technology.

9. **Second-order effect.** Let $a_{1t} = \phi S + \varepsilon$ and $a_{2t} = (1-\phi)S - \varepsilon$. Show that

$$\ln A_t - \ln A^* \approx -\frac{\theta}{1-\theta}\,\frac{\varepsilon^2}{2S^2\phi(1-\phi)},$$

where A^* is the value with equal spreads. Why is the first-order term zero?

Answer. $\ln G = \phi \ln(\phi S + \varepsilon) + (1-\phi)\ln((1-\phi)S - \varepsilon)$. First-order term: $\phi \cdot \frac{\varepsilon}{\phi S} - (1-\phi)\frac{\varepsilon}{(1-\phi)S} = 0$. Second-order term: $-\frac{\varepsilon^2}{2S^2}\left(\frac{1}{\phi} + \frac{1}{1-\phi}\right) = -\frac{\varepsilon^2}{2S^2\phi(1-\phi)}$. Multiply by $\theta/(1-\theta)$. The first-order term vanishes because the equal-spread allocation is the maximum (Question 8): starting from an efficient allocation, small misallocations have second-order costs (Harberger triangle). Starting from an already distorted allocation, the effect would be first order.

10. A numerical example. Let $\phi = \theta = 1/2$. In period 0, $\tau_{10} = \tau_{20} = 20\%$.

(a) Compute S and the labor wedge τ_l in period 0.

(b) In period 1, small firms are hit by a credit crunch: $\tau_{11} = 40\%$. Find τ_{21} such that S is unchanged. Compute the percentage change in A_t and in τ_{lt}. Compare with the approximation of Question 9.

(c) Suppose instead that both spreads rise to 40%. Compute the change in A_t and in τ_{lt}.

Answer. (a) $a_1 = a_2 = 0.5/1.2 = 0.4167$, $S = 0.8333$. $1-\tau_l = (1-\theta)/(1-\theta S) = 0.5/0.5833 = 0.857$, so $\tau_l = 14.3\%$. (b) $a_{11} = 0.5/1.4 = 0.3571$, so $a_{21} = 0.8333 - 0.3571 = 0.4762$ and $1+\tau_{21} = 0.5/0.4762 = 1.05$: $\tau_{21} = 5\%$. With $\theta/(1-\theta) = 1$ and $\phi = 1/2$, $A_t \propto \sqrt{a_1 a_2}$: $\ln A_1 - \ln A_0 = \ln\left(\sqrt{0.3571 \times 0.4762}/0.4167\right) = \ln(0.4124/0.4167) \approx -1.03\%$. The labor wedge is unchanged at 14.3%. Approximation: $\varepsilon = -0.0595$, so $-\varepsilon^2/(2 \times 0.8333^2 \times 0.25) \approx -1.02\%$, very close. (c) $a_1 = a_2 = 0.3571$, $S = 0.7143$. With equal spreads $A_t \propto (S/2)(1-S/2)$: $\ln\left[(0.3571 \times 0.6429)/(0.4167 \times 0.5833)\right] = \ln(0.2296/0.2431) \approx -5.7\%$. The labor wedge rises: $1-\tau_l = 0.5/0.6429 = 0.778$, $\tau_l = 22.2\%$. A general tightening of credit moves *both* the efficiency and the labor (and capital) wedges.

11. Discussion. An econometrician observes data generated by the economy in Question 10(b) and measures the wedges of the prototype economy.

(a) What would she conclude about the source of the recession?

(b) What additional data would allow her to distinguish a financial shock from a genuine technology shock?

(c) What does this example imply for the interpretation of the Solow residual in the RBC literature?

Answer. (a) She would measure a fall in A_t of about 1% with constant labor and investment wedges, and attribute the recession to technological regress, even though no technology changed. (b) Sectoral or firm-level data: credit spreads by firm size, the use of intermediate inputs relative to gross output by sector, dispersion of marginal products of intermediates across firms. In the model, $m_{1t}/(p_{1t}q_{1t}) = \theta/(1+\tau_{1t})$ falls in the constrained sector and rises in the other. (c) The Solow residual is a reduced form: it includes misallocation and financial frictions (this connects to Lecture 6 on misallocation, Hsieh & Klenow 2009). BCA identifies the *wedge* that matters, not the primitive shock: finding that A_t drives a recession calls for models in which frictions show up as TFP, not necessarily for technology shocks.

Exercise 2 – Sticky wages → labor wedge

The detailed economy (Bordo, Erceg & Evans, 2000). Let s^t denote the history of events and $\pi_t(s^t)$ its probability.

- A competitive final-good producer produces $y(s^t) = F\big(k(s^{t-1}), l(s^t)\big)$ with a labor aggregate of differentiated labor types $j \in [0,1]$:

$$l(s^t) = \left[\int_0^1 l(j,s^t)^v \, dj\right]^{1/v}, \qquad v \in (0,1).$$

It accumulates capital, $k(s^t) = (1-\delta)k(s^{t-1}) + x(s^t)$, and maximizes $\sum_t \sum_{s^t} Q(s^t)\big[P(s^t)y(s^t) - P(s^t)x(s^t) - W(s^{t-1})l(s^t)\big]$, where $Q(s^t)$ is the price of a dollar at s^t, $P(s^t)$ the price level and $W(s^{t-1})$ the aggregate nominal wage.

- Consumers are organized in unions; union j supplies labor of type j. It sets its nominal wage $W(j, s^{t-1})$ *one period in advance*, before the realization of s_t, and then supplies whatever labor is demanded. Members have preferences

$$\sum_t \sum_{s^t} \beta^t \pi_t(s^t) \Big[U\big(c(j,s^t), l(j,s^t)\big) + V\big(M(j,s^t)/P(s^t)\big)\Big],$$

trade a complete set of state-contingent nominal bonds, and receive dividends and lump-sum money transfers $P(s^t)T(s^t) = M(s^t) - M(s^{t-1})$.

- Money supply: $M(s^t) = \mu(s^t)M(s^{t-1})$, where $\mu(s^t)$ is a stochastic monetary shock.

Part A – Labor demand and wage setting

1. Demand for each labor type. For a given aggregate $l(s^t)$, the producer minimizes $\int_0^1 W(j,s^{t-1})\, l(j,s^t)\, dj$ subject to the definition of $l(s^t)$. Show that

$$l^d(j,s^t) = \left[\frac{W(s^{t-1})}{W(j,s^{t-1})}\right]^{\frac{1}{1-v}} l(s^t), \qquad W(s^{t-1}) = \left[\int_0^1 W(j,s^{t-1})^{\frac{v}{v-1}} dj\right]^{\frac{v-1}{v}},$$

and that the minimized cost is $W(s^{t-1})\, l(s^t)$. What is the elasticity of demand for labor type j with respect to its wage?

Answer. Lagrangian with multiplier Λ: FOC $W(j) = \Lambda\, l^{1-v} l(j)^{v-1}$, so $l(j) = (W(j)/\Lambda)^{\frac{1}{v-1}} l$. Plugging into the aggregator, $1 = \Lambda^{\frac{v}{1-v}} \int W(j)^{\frac{v}{v-1}} dj$, so $\Lambda = W$ as defined. Then $l(j) = (W/W(j))^{\frac{1}{1-v}} l$ and $\int W(j) l(j) dj = W^{\frac{1}{1-v}} l \int W(j)^{\frac{v}{v-1}} dj = Wl$. The elasticity is $-\frac{1}{1-v} < -1$; it goes to $-\infty$ as $v \to 1$ (perfect substitutes).

2. The producer's first-order conditions. Show that the producer's choice of aggregate labor and capital implies

$$P(s^t)\, F_l(s^t) = W(s^{t-1}), \qquad Q(s^t)P(s^t) = \sum_{s_{t+1}} Q(s^{t+1})P(s^{t+1})\big[F_k(s^{t+1}) + 1 - \delta\big].$$

Explain why, given $W(s^{t-1})$, a surprise increase in $P(s^t)$ raises employment.

Answer. Substitute $x(s^t) = k(s^t) - (1-\delta)k(s^{t-1})$ into profits. FOC for $l(s^t)$: $Q(s^t)[P(s^t)F_l(s^t) - W(s^{t-1})] = 0$. FOC for $k(s^t)$: $-Q(s^t)P(s^t) + \sum_{s_{t+1}} Q(s^{t+1})P(s^{t+1})[F_k(s^{t+1}) + 1 - \delta] = 0$. The real wage is $W(s^{t-1})/P(s^t)$: with the nominal wage pre-set, a higher price level lowers the real wage, and with $F_{ll} < 0$ the producer hires more labor until F_l falls to the new real wage.

3. Wage setting. From the bond FOC, $Q(s^t) \propto \beta^t \pi_t(s^t)\, U_c(s^t)/P(s^t)$. Use this to show that, when choosing $W(j, s^{t-1})$, union j maximizes

$$\sum_{s^t | s^{t-1}} Q(s^t) \left[W(j, s^{t-1})\, l^d(j, s^t) + P(s^t) \frac{U_l(s^t)}{U_c(s^t)}\, l^d(j, s^t) \right],$$

taking aggregates as given. Derive the optimal wage

$$W(j, s^{t-1}) = -\frac{\sum_{s^t} Q(s^t) P(s^t) \frac{U_l(j, s^t)}{U_c(j, s^t)}\, l^d(j, s^t)}{v \sum_{s^t} Q(s^t)\, l^d(j, s^t)}.$$

Answer. Wage income $W(j) l^d(j)$ is worth $\beta^t \pi_t U_c / P \propto Q$ in utility; the disutility of work $U_l\, l^d(j)$, converted into dollars by dividing by U_c/P, is $P(U_l/U_c) l^d(j)$, also weighted by Q. Using $\partial l^d(j)/\partial W(j) = -\frac{1}{1-v} \frac{l^d(j)}{W(j)}$, the FOC is

$$\sum Q \Big[l^d(j) - \tfrac{1}{1-v} l^d(j) - \tfrac{1}{1-v} \tfrac{P}{W(j)} \tfrac{U_l}{U_c} l^d(j) \Big] = 0 \iff \sum Q \Big[v\, l^d(j) + \tfrac{P}{W(j)} \tfrac{U_l}{U_c} l^d(j) \Big] = 0,$$

which rearranges to the stated formula. The wage is a weighted average, across the states s^t that can follow s^{t-1}, of the nominal marginal rate of substitution, marked up by $1/v$.

4. Steady state. Show that in a deterministic steady state $\dfrac{W}{P} = \dfrac{1}{v}\left(-\dfrac{U_l}{U_c}\right)$. Combine this with Question 2 to compute the ratio of the marginal rate of substitution (MRS) to the marginal product of labor (MPL) in steady state. Give a value for $v = 0.9$.

Answer. In a steady state, Q, P, U_l/U_c and l^d do not depend on s^t (up to a common scale), so the sums cancel. With $W/P = F_l$: MRS/MPL $= (-U_l/U_c)/F_l = v$. For $v = 0.9$, the real wage is an 11% markup over the MRS, and the MRS is 90% of the MPL: union monopoly power acts like a 10% tax on labor.

Part B – The equivalent real economy

Consider a *real* prototype economy with the same technology and producer problem, in which a representative consumer maximizes $\sum_t \sum_{s^t} \beta^t \pi_t(s^t) U(c(s^t), l(s^t))$ subject to

$$c(s^t) + \sum_{s_{t+1}} q(s^{t+1}|s^t)\, b(s^{t+1}) \le \big[1 - \tau_l(s^t)\big] w(s^t) l(s^t) + b(s^t) + \text{transfers} + \text{dividends},$$

where lowercase letters are real variables.

5. Proposition 2. Write down the consumer's labor FOC in the prototype economy. Let starred variables denote the equilibrium of the sticky-wage economy for a given money supply process $M^*(s^t)$. Show that if

$$1 - \tau_l(s^t) = -\frac{U_l^*(s^t)}{U_c^*(s^t)} \frac{1}{F_l^*(s^t)},$$

and real prices are set to $w(s^t) = W^*(s^{t-1})/P^*(s^t)$ and $q(s^{t+1}|s^t) = Q^*(s^{t+1}) P^*(s^{t+1}) / [Q^*(s^t) P^*(s^t)]$, then the starred allocations satisfy all equilibrium conditions of the prototype economy.

Answer. Labor FOC: $-U_l/U_c = (1 - \tau_l) w$. Check each condition. (i) Firms: $w = F_l$ is the starred condition $P^* F_l^* = W^*$. (ii) Labor: $(1 - \tau_l) w = (-U_l^*/U_c^*)(1/F_l^*) \cdot F_l^* = -U_l^*/U_c^*$, so

the starred allocation satisfies the prototype FOC by construction. (iii) Bonds and capital: the real bond price is the nominal one adjusted for inflation, so the bond FOC and the capital Euler equation are the starred ones expressed in real terms. (iv) The resource constraint $c + k(s^t) = F(k(s^{t-1}), l(s^t)) + (1-\delta)k(s^{t-1})$ is identical. Transfers are set equal to the real value of transfers in the sticky-wage economy adjusted for the interest cost of holding money, so budget constraints hold. Money has no other role: it only matters through the wedge it creates between the MRS and the MPL.

6. The steady-state labor wedge. Using Question 4, what is the value of τ_l in the steady state? Is the labor wedge always positive in this economy?

Answer. $1 - \tau_l = \text{MRS}/\text{MPL} = v$, so $\tau_l = 1 - v$. The labor wedge fluctuates around $1 - v > 0$ but need not be positive in every state: after a large positive monetary surprise, employment can rise so much that MRS > MPL and $\tau_l(s^t) < 0$.

Part C – Monetary shocks as labor wedges

Let $F(k, l) = k^\alpha l^{1-\alpha}$ and $U(c, l) = \ln c + \psi \ln(1 - l)$. Consider a period-$t$ monetary surprise that raises the price level by $d \ln P_t > 0$. Capital $k(s^{t-1})$ and the nominal wage $W(s^{t-1})$ are pre-determined.

7. Employment and output. Using $P(s^t)F_l(s^t) = W(s^{t-1})$, show that $d \ln l_t = d \ln P_t/\alpha$ and compute $d \ln y_t$. Give numbers for $\alpha = 1/3$ and $d \ln P_t = 1\%$.

Answer. $F_l = (1-\alpha)(k/l)^\alpha$, so $d \ln F_l = -\alpha\, d \ln l$ with k fixed. From $d \ln P + d \ln F_l = 0$: $d \ln l = d \ln P/\alpha$. Then $d \ln y = (1-\alpha) d \ln l = \frac{1-\alpha}{\alpha} d \ln P$. With $\alpha = 1/3$: $d \ln l = 3\%$ and $d \ln y = 2\%$.

8. Response of the measured labor wedge. Show that

$$d \ln(1 - \tau_{lt}) = d \ln c_t + \left(\frac{l}{1-l} + \alpha \right) d \ln l_t.$$

Evaluate it with $l = 1/3$, $\alpha = 1/3$, $d \ln l_t = 3\%$ and $d \ln c_t = 1\%$. Does the labor wedge rise or fall after a monetary expansion? Explain intuitively using the gap between MRS and MPL.

Answer. $1 - \tau_l = \text{MRS}/\text{MPL}$ with $\text{MRS} = \psi c/(1-l)$ and $\text{MPL} = (1-\alpha)y/l$. $d \ln \text{MRS} = d \ln c + \frac{l}{1-l} d \ln l$ and $d \ln \text{MPL} = -\alpha\, d \ln l$. Subtracting gives the formula. Numbers: $1\% + (0.5 + 0.333) \times 3\% = 3.5\%$. So $1 - \tau_l$ rises, i.e. the labor "tax" τ_l *falls*. Intuition: the real wage falls below its desired level, firms move down their labor demand curve, so the MPL falls, while households work more and the MRS rises. The gap between MRS and MPL created by the union markup narrows. A monetary expansion looks like a labor-tax cut in the prototype economy.

9. The Great Depression. Bordo, Erceg & Evans (2000) argue that the monetary contraction of 1929–33 caused the Depression through sticky wages. What does the model predict for the measured labor wedge in that episode? CKM find that the labor wedge worsened sharply in 1930–33 but that the efficiency wedge accounts for most of the fall in output. What does this suggest about the sticky-wage mechanism as the *only* explanation?

Answer. A contraction lowers P with W pre-set: the real wage is too high, employment falls, the MPL rises and the MRS falls, so $1 - \tau_l$ falls (the labor wedge worsens). This is consistent with the measured labor wedge. However, the sticky-wage model with monetary shocks generates movements

in the labor wedge only; it cannot generate the large fall in measured TFP. The BCA results point to models in which the primitive shocks also show up as efficiency wedges (e.g. financial frictions as in Exercise 1).

10. Persistence. In this model, wages are set only one period in advance. How long does the effect of a one-time monetary surprise on the labor wedge last? Compare with the persistence of measured labor wedges in the data. What modifications of the model would be needed?

Answer. One period: in $t+1$, unions reset wages taking the new price level into account, and the wedge returns to (around) $1-v$. Measured labor wedges are very persistent (they move with low-frequency swings in hours, e.g. the slow recovery of the 1930s). One needs staggered or longer wage contracts (Taylor or Calvo), persistent monetary shocks, or other persistent sources of labor wedges (unions and cartel policies under the NIRA, Cole & Ohanian 2004; search frictions; labor taxes). See also Chari, Kehoe & McGrattan (2000) on the persistence problem in sticky-price models.

11. Lucas critique. Suppose the central bank changes its money supply rule $\mu(s^t)$. Is the stochastic process for $\tau_l(s^t)$ in the equivalent prototype economy unchanged? What does this imply for using BCA to evaluate counterfactual policies?

Answer. No. The wedge is an equilibrium object of the detailed economy: it depends on the realized monetary shocks *and* on how unions forecast prices when setting wages, hence on the policy rule. A change in the rule changes the joint process of wedges. BCA is an accounting device to select promising classes of models, not a structural model for policy evaluation: counterfactual policy analysis requires writing down and estimating the detailed economy.

Exercise 3 – Markup shocks → labor and investment wedges

The detailed economy. Time is discrete, s^t is the history of shocks and $\Pi_t(s^t)$ its probability. There is no rental market for capital; each firm owns its own.

- A representative household solves

$$\max_{\{C_t(s^t),H_t(s^t)\}} \sum_{t=0}^{\infty}\beta^t \sum_{s^t}\Pi_t(s^t)\,U\big(C_t(s^t),H_t(s^t)\big)$$

$$\text{subject to} \qquad A_0=\sum_{t=0}^{\infty}\sum_{s^t}Q_0^t(s^t)\big[P_t(s^t)C_t(s^t)-W_t(s^t)H_t(s^t)\big],$$

  taking as given the date-0 nominal prices $Q_0^t(s^t)$, the wage $W_t(s^t)$ and the price level $P_t(s^t)$.

- A competitive final-good producer bundles a continuum of intermediate varieties $j\in[0,1]$,

$$Y_t(s^t)=\left[\int_0^1 y_{jt}(s^t)^{\frac{\varepsilon_t(s^t)-1}{\varepsilon_t(s^t)}}\,dj\right]^{\frac{\varepsilon_t(s^t)}{\varepsilon_t(s^t)-1}}, \qquad \varepsilon_t(s^t)>1,$$

  and maximizes $\sum_t\sum_{s^t}Q_0^t\big[P_tY_t-\int_0^1 p_{jt}\,y_{jt}\,dj\big]$ taking P_t and $\{p_{jt}\}$ as given. The elasticity of substitution ε_t is stochastic and is the only source of uncertainty.

- Intermediate producer j chooses its price, labor and capital to maximize

$$V_{j0} = \max_{\{p_{jt},h_{jt},k_{j,t+1}\}} \sum_{t=0}^{\infty} \sum_{s^t} Q_0^t(s^t)\big[p_{jt}\, y_{jt} - W_t\, h_{jt} - P_t\, x_{jt}\big], \qquad y_{jt} = k_{jt}^{\alpha}\big(z_t\, h_{jt}\big)^{1-\alpha},$$

  with $k_{j,t+1} = (1-\delta)k_{jt} + x_{jt}$, $k_{j0} = k_0$ given, taking as given Q_0^t, W_t, P_t and its own demand curve $y_{jt} = \Phi_t(p_{jt}, s^t)$.

- Market clearing: $H_t = \int_0^1 h_{jt}\, dj$ and $Y_t = C_t + \int_0^1 x_{jt}\, dj$.

Throughout, define the gross markup

$$\mu_t \equiv \frac{\varepsilon_t}{\varepsilon_t - 1} > 1.$$

Part A – Solving the detailed economy

1. Household. Derive the first-order conditions. Show that they imply a labor-supply condition and an expression for the nominal stochastic discount factor:

$$-\frac{U_{Ht}}{U_{Ct}} = \frac{W_t}{P_t}, \qquad \frac{Q_0^{t+1}(s^{t+1})\, P_{t+1}(s^{t+1})}{Q_0^t(s^t)\, P_t(s^t)} = \beta\, \frac{\Pi_{t+1}(s^{t+1})}{\Pi_t(s^t)}\, \frac{U_{Ct+1}(s^{t+1})}{U_{Ct}(s^t)}.$$

Answer. With multiplier λ on the single lifetime budget constraint, $\beta^t \Pi_t U_{Ct} = \lambda\, Q_0^t P_t$ and $-\beta^t \Pi_t U_{Ht} = \lambda\, Q_0^t W_t$. Dividing the second by the first gives the labor-supply condition. Taking the ratio of the first condition at s^{t+1} and s^t gives $\frac{Q_0^{t+1} P_{t+1}}{Q_0^t P_t} = \beta \frac{\Pi_{t+1}}{\Pi_t} \frac{U_{Ct+1}}{U_{Ct}}$. Note that only *one* multiplier appears: with complete markets the household's problem is static in the Arrow-Debreu sense.

2. Final-good producer. Derive the first-order condition. Show that the problem has a solution only if

$$P_t = \left[\int_0^1 p_{jt}^{1-\varepsilon_t}\, dj\right]^{\frac{1}{1-\varepsilon_t}},$$

that profits are then zero, and that the demand curve faced by firm j is

$$\Phi_t(p_{jt}) = \left(\frac{p_{jt}}{P_t}\right)^{-\varepsilon_t} Y_t.$$

Answer. The FOC with respect to y_{jt} is $P_t\, \partial Y_t / \partial y_{jt} = p_{jt}$, i.e. $P_t (Y_t/y_{jt})^{1/\varepsilon_t} = p_{jt}$, which already gives the demand curve once P_t is known; comparing two varieties, $(y_{jt}/y_{0t})^{-1/\varepsilon_t} = p_{jt}/p_{0t}$. The aggregator is homogeneous of degree one in the y_{jt}, so if the producer made a positive profit at some scale it could scale up without bound: in equilibrium profits must be zero, $P_t Y_t = \int p_{jt} y_{jt}\, dj$. Substituting the demand curve into this zero-profit condition gives the ideal price index above, and hence $W_0 = 0$ and $y_{jt} = (p_{jt}/P_t)^{-\varepsilon_t} Y_t$.

3. Intermediate producer. Write the firm's problem with the demand curve substituted in and a multiplier λ_{jt} on the production function. Show that

$$\underbrace{\lambda_{jt} = \frac{W_t\, h_{jt}^{\alpha}}{(1-\alpha)\, k_{jt}^{\alpha}\, z_t^{1-\alpha}}}_{\text{nominal marginal cost}}, \qquad p_{jt} = \mu_t\, \lambda_{jt},$$

$$Q_0^t P_t = \sum_{s^{t+1} \succ s^t} Q_0^{t+1} \left[P_{t+1}(1-\delta) + \lambda_{j,t+1}\, \alpha\, k_{j,t+1}^{\alpha-1} \big(z_{t+1} h_{j,t+1}\big)^{1-\alpha} \right].$$

Answer. The FOC for h_{jt} gives $W_t = \lambda_{jt}(1-\alpha) k_{jt}^{\alpha} z_t^{1-\alpha} h_{jt}^{-\alpha} = \lambda_{jt}\, \partial y_{jt}/\partial h_{jt}$, i.e. λ_{jt} is the nominal marginal cost (the wage divided by the marginal product of labor). The FOC for p_{jt} is

$$(1-\varepsilon_t) p_{jt}^{-\varepsilon_t} P_t^{\varepsilon_t} Y_t + \lambda_{jt}\, \varepsilon_t\, p_{jt}^{-\varepsilon_t - 1} P_t^{\varepsilon_t} Y_t = 0 \iff p_{jt} = \frac{\varepsilon_t}{\varepsilon_t - 1} \lambda_{jt} = \mu_t \lambda_{jt},$$

the usual constant-markup rule. The FOC for $k_{j,t+1}$ equates the nominal cost of a unit of capital, $Q_0^t P_t$, to the discounted value of the undepreciated capital plus the marginal product of capital valued at marginal cost.

4. **Aggregate equilibrium conditions.** Impose symmetry ($p_{jt} = P_t$, $h_{jt} = H_t$, $k_{jt} = K_t$) and combine Questions 1–3 to show that the equilibrium is characterized by the resource constraint, $Y_t = K_t^{\alpha} (z_t H_t)^{1-\alpha}$, and

$$-\frac{U_{Ht}}{U_{Ct}} = \frac{1}{\mu_t} \underbrace{(1-\alpha)\frac{Y_t}{H_t}}_{=\partial Y_t/\partial H_t}, \qquad U_{Ct} = \beta\, \mathbb{E}_t \left[U_{Ct+1} \left\{ \frac{1}{\mu_{t+1}}\, \alpha \left(\frac{z_{t+1} H_{t+1}}{K_{t+1}} \right)^{1-\alpha} + 1 - \delta \right\} \right].$$

Answer. Symmetry and $p_{jt} = \mu_t \lambda_{jt}$ give $\lambda_{jt} = P_t/\mu_t$: real marginal cost equals $1/\mu_t$. Substituting into the labor FOC, $W_t/P_t = (1/\mu_t)(1-\alpha) K_t^{\alpha} z_t^{1-\alpha} H_t^{-\alpha} = (1/\mu_t)\partial Y_t/\partial H_t$, and the household's labor-supply condition delivers the first equation. Substituting $\lambda_{j,t+1} = P_{t+1}/\mu_{t+1}$ into the capital FOC,

$$Q_0^t P_t = \sum_{s^{t+1}} Q_0^{t+1} P_{t+1} \left[1 - \delta + \frac{1}{\mu_{t+1}} \alpha \Big(\frac{z_{t+1} H_{t+1}}{K_{t+1}} \Big)^{1-\alpha} \right],$$

and replacing $Q_0^{t+1} P_{t+1}/(Q_0^t P_t)$ by the household's SDF from Question 1 gives the Euler equation (the sum over s^{t+1} of Π_{t+1}/Π_t times a variable is $\mathbb{E}_t$ of that variable).

Part B – Which wedges do markup shocks generate?

5. **Efficiency wedge.** Show that the aggregate production function of the detailed economy is that of the prototype economy with $A_t = z_t^{1-\alpha}$ and $F(K,H) = K^{\alpha} H^{1-\alpha}$. Does a markup shock move the efficiency wedge? Compare with Exercise 1.

   Answer. In the symmetric equilibrium all firms charge the same price and hence produce the same quantity, so there is no dispersion across varieties and $Y_t = \int y_{jt} dj = K_t^{\alpha} (z_t H_t)^{1-\alpha} = z_t^{1-\alpha} K_t^{\alpha} H_t^{1-\alpha}$. Measured TFP moves only with true technology z_t. This is in sharp contrast with Exercise 1: there, the friction distorted the *allocation* of an input across sectors and therefore showed up in A_t; here, the markup is common to all firms, so nothing is misallocated *across* firms and the only distortion is in the *level* of inputs. *Remark:* with heterogeneous or staggered markups (e.g. Calvo pricing, Exercise 5) relative prices *do* differ across firms and the resulting dispersion *does* generate an efficiency wedge.

6. **Labor and capital wedges.** Consider a prototype economy in which the household faces a tax τ_{lt} on labor income and τ_{kt} on capital income, and firms pay factors their marginal products. Show that the two economies have the same first-order conditions if and only if

$$1 - \tau_{lt} = 1 - \tau_{kt} = \frac{1}{\mu_t}.$$

Compare with Question 5 of Exercise 1. Why is the distortion the same for labor and capital?

Answer. The detailed economy pays $W_t/P_t = (1/\mu_t)F_{Ht}$ and, implicitly, a return on capital $(1/\mu_t)A_tF_{Kt}$; the prototype pays $(1-\tau_{lt})A_tF_{Ht}$ and $(1-\tau_{kt})A_tF_{Kt}$. Matching term by term gives the result, so $\tau_{kt} = \tau_{lt} = 1 - 1/\mu_t$. This is exactly the structure of Exercise 1: there the wedge was v_t/A_t, here it is $1/\mu_t$, and in both cases it multiplies the marginal product of *every* factor. The reason is the same: the distortion operates on the *output* side (a gap between the price of the good and its marginal cost) rather than on one particular input. Since both factors are remunerated out of the same revenue, a given wedge between price and marginal cost scales both marginal revenue products down by the same factor. In this sense, input-financing frictions, markups and labor taxes are all observationally equivalent at the level of the wedges – which is precisely CKM's point.

7. The investment wedge. CKM's prototype economy has an investment wedge $1+\tau_{xt}$ instead of a tax on capital income:

$$U_{Ct}\,(1+\tau_{xt}) = \beta\,\mathbb{E}_t\Big[U_{Ct+1}\big\{A_{t+1}F_{Kt+1} + (1-\delta)(1+\tau_{xt+1})\big\}\Big].$$

(a) Show that with full depreciation ($\delta = 1$) and μ_{t+1} known at t, the markup economy is a prototype economy with $1+\tau_{xt} = \mu_{t+1}$.

(b) Show that with $\delta < 1$ the two economies are equivalent only if the markup is constant. Where does the difference come from?

Answer. (a) With $\delta = 1$ the markup Euler equation is $U_{Ct} = \beta\mathbb{E}_t[U_{Ct+1}A_{t+1}F_{Kt+1}/\mu_{t+1}]$ and the prototype one is $U_{Ct}(1+\tau_{xt}) = \beta\mathbb{E}_t[U_{Ct+1}A_{t+1}F_{Kt+1}]$; if μ_{t+1} is known at t it can be taken out of the expectation and $1+\tau_{xt} = \mu_{t+1}$ does the job. (b) Write $1+\tau_{xt} = \Theta_t$ and divide the prototype Euler equation by Θ_t: $U_{Ct} = \beta\mathbb{E}_t[U_{Ct+1}\{A_{t+1}F_{Kt+1}/\Theta_t + (1-\delta)\Theta_{t+1}/\Theta_t\}]$. Matching with the markup economy requires $\Theta_t = \mu_{t+1}$ *and* $\Theta_{t+1} = \Theta_t$, i.e. a constant markup. The reason: the markup distorts only the *marginal product* of capital, whereas an investment wedge also taxes the resale of the undepreciated stock $(1-\delta)k$. With $\delta = 1$ there is no undepreciated stock and the two coincide. In practice, BCA exercises report the investment wedge, so a markup shock is read as a *combination* of labor and investment wedges.

Part C – Calibrating the balanced growth path

Assume $z_t = (1+g)^t$ and $\varepsilon_t = \varepsilon$ constant, and take

$$U(C,H) = \ln C - \gamma\,\frac{H^{1+\varphi}}{1+\varphi},$$

where φ is the inverse Frisch elasticity of labor supply, as in Lecture 2. Look for a balanced growth path with $C_t = (1+g)^t c^*$, $K_t = (1+g)^t k^*$, $H_t = h^*$.

8. Factor shares. Show that on the balanced growth path

$$\underbrace{s_L \equiv \frac{Wh^*}{Py^*} = \frac{1-\alpha}{\mu}}_{\text{labor share}}, \qquad \underbrace{s_K = \frac{\alpha}{\mu}}_{\text{capital income share}}, \qquad \underbrace{s_\pi = 1 - \frac{1}{\mu}}_{\text{pure profit share}},$$

and that these three shares add up to one.

Answer. Labor is paid $(1/\mu)(1-\alpha)y/h$, so its share of revenue is $(1-\alpha)/\mu$; capital earns $(1/\mu)\alpha y/k$ per unit, so its share is α/μ. The two factor shares sum to $1/\mu$, so the residual, which accrues to the owners of the firms as pure profit, is $1 - 1/\mu = 1/\varepsilon$.

9. Balanced growth path. Show that (c^*, k^*, h^*) solve

$$\gamma\, c^*\, (h^*)^{\varphi} = \frac{1}{\mu}\frac{(1-\alpha)(k^*)^{\alpha}}{(h^*)^{\alpha}}, \qquad 1+g = \beta\left[1-\delta+\frac{\alpha}{\mu}\Big(\frac{h^*}{k^*}\Big)^{1-\alpha}\right], \qquad (k^*)^{\alpha}(h^*)^{1-\alpha} = c^*+(g+\delta)k^*.$$

Answer. Substitute the conjectured growth rates into the three equilibrium conditions of Question 4. In the labor-supply condition the factors $(1+g)^t$ cancel on both sides; in the Euler equation $U_{Ct} = 1/C_t$ makes the left-hand side $(1+g)^{-t}/c^*$ and the right-hand side $\beta(1+g)^{-(t+1)}/c^*$ times the gross return, so the growth factor $(1+g)$ appears on the left; in the resource constraint $(1+g)k_{t+1}-(1-\delta)k_t$ becomes $(g+\delta)k^*$ in units of the trend.

10. Calibration. Take as quarterly targets $s_L = 0.60$, $K/Y = 12.8$, $X/Y = 0.26$ and $g = 0.005$.

(a) Compute δ and c^*/y^*. Do they depend on μ?

(b) Show that $\alpha = 1-\mu\, s_L$ and that the gross return on capital is

$$\frac{\alpha}{\mu}\frac{Y}{K} = \Big(\frac{1}{\mu}-s_L\Big)\frac{Y}{K}.$$

Compute α, s_π and β for $\varepsilon \to \infty$, $\varepsilon = 11$ and $\varepsilon = 6$. Comment.

(c) Normalizing $h^* = 1$, show that $\gamma = s_L/(c^*/y^*)$ and compute it. Does it depend on μ?

(d) How does the markup change the calibration of the balanced growth path? Which additional data would identify μ?

Answer. (a) $X/Y = (g+\delta)K/Y$ gives $g+\delta = 0.26/12.8 = 0.0203$, so $\delta = 0.0153$ (about 6% a year); and $c^*/y^* = 1-0.26 = 0.74$. Neither uses the markup: they come from the resource constraint and the capital accumulation identity only. (b) $s_L = (1-\alpha)/\mu$ gives $\alpha = 1-\mu s_L$, hence $\alpha/\mu = 1/\mu - s_L$, which is *decreasing* in μ. So a larger markup raises the calibrated α but *lowers* the measured return on capital, and $\beta = (1+g)/[1-\delta+(1/\mu-s_L)(Y/K)]$ must be larger:

	μ	α	s_π	return $\alpha Y/(\mu K)$	β
$\varepsilon \to \infty$	1.000	0.400	0%	0.0313	0.9892
$\varepsilon = 11$	1.100	0.340	9.1%	0.0241	0.9962
$\varepsilon = 6$	1.200	0.280	16.7%	0.0182	1.0021

With $\varepsilon = 6$ the net real return is $0.0182 - 0.0153 = 0.0029$ per quarter, i.e. 1.2% a year, *below* the growth rate of 2% a year: the model can then only be reconciled with the data if $\beta > 1$. The markup, the capital share and the discount factor are therefore jointly constrained by the data – this is the heart of the recent debate on rising markups and the measurement of capital income (De Loecker, Eeckhout & Unger 2020; Barkai 2020; Karabarbounis & Neiman 2019). (c) Combining the labor-supply condition with $s_L = (1/\mu)(1-\alpha) = Wh/(Py)$ gives $\gamma c^*(h^*)^{1+\varphi} = s_L y^*$, so with $h^* = 1$, $\gamma = s_L/(c^*/y^*) = 0.60/0.74 = 0.811$, independent of μ and of φ (the markup has already been absorbed in the labor-share target). (d) The markup is *not* separately identified from the labor share: only the product $\mu s_L = 1-\alpha$ is. Given s_L, assuming competition ($\mu = 1$) and assuming $\mu = 1.2$ deliver the same labor share but capital shares that differ by 12 percentage points and discount factors that differ by 1.3% per quarter. To identify μ one needs an extra observable: the profit share $s_\pi = 1-1/\mu$, micro price-cost margins, or the pass-through of marginal cost into prices.

Part D – Markup shocks and the business cycle

Assume now $\log \varepsilon_t$ follows a first-order Markov process with mean $\log \bar{\varepsilon}$ and autocorrelation ρ. Write $\widehat{v}_t \equiv \log(v_t/v^*)$ for log deviations of detrended variables.

11. From ε to μ. Show that $\widehat{\mu}_t = -\dfrac{\widehat{\varepsilon}_t}{\varepsilon - 1}$ and that $\widehat{\mu}_t$ inherits the autocorrelation ρ. For $\varepsilon = 11$, by how much does a 1% rise in ε move the markup and the measured labor wedge?

Answer. $\log \mu = \log \varepsilon - \log(\varepsilon - 1)$, so $d\log\mu / d\log\varepsilon = 1 - \varepsilon/(\varepsilon-1) = -1/(\varepsilon-1)$. Being a linear function of $\widehat{\varepsilon}_t$ (to first order), $\widehat{\mu}_t$ is an AR(1) with the same persistence. With $\varepsilon = 11$, a 1% rise in ε lowers the markup by 0.1%. Since $1 - \tau_{lt} = 1/\mu_t$, we have $d\tau_{lt} = (1-\tau_{lt})\widehat{\mu}_t = \widehat{\mu}_t/\mu$: the labor wedge moves by 0.09 percentage points. Markup shocks are a *small-leverage* source of wedge movements unless ε is close to 1.

12. Log-linearized system. Set $g = 0$ for simplicity and let $R \equiv 1 - \delta + \rho_k$ with $\rho_k \equiv (\alpha/\mu)(h^*/k^*)^{1-\alpha}$. Show that the equilibrium is characterized by

$$\widehat{c}_t + (\varphi + \alpha)\widehat{h}_t = \alpha \widehat{k}_t - \widehat{\mu}_t \qquad \text{(labor wedge)}$$

$$\mathbb{E}_t \widehat{c}_{t+1} - \widehat{c}_t = \frac{\rho_k}{R}\, \mathbb{E}_t\big[(1-\alpha)(\widehat{h}_{t+1} - \widehat{k}_{t+1}) - \widehat{\mu}_{t+1}\big] \qquad \text{(Euler)}$$

$$\alpha \widehat{k}_t + (1-\alpha)\widehat{h}_t = \frac{c^*}{y^*}\widehat{c}_t + \frac{k^*}{y^*}\big[\widehat{k}_{t+1} - (1-\delta)\widehat{k}_t\big] \qquad \text{(resource)}$$

Answer. Take logs of the three conditions of Question 10 and differentiate. The labor-supply condition $\gamma c_t h_t^{\varphi} = (1/\mu_t)(1-\alpha)k_t^{\alpha} h_t^{-\alpha}$ gives $\widehat{c}_t + \varphi \widehat{h}_t = -\widehat{\mu}_t + \alpha\widehat{k}_t - \alpha\widehat{h}_t$. For the Euler equation, $1/c_t = \beta \mathbb{E}_t[(1/c_{t+1}) R_{t+1}]$ with $R_{t+1} = 1 - \delta + (\alpha/\mu_{t+1})(h_{t+1}/k_{t+1})^{1-\alpha}$: to first order $-\widehat{c}_t = -\mathbb{E}_t\widehat{c}_{t+1} + \mathbb{E}_t \widehat{R}_{t+1}$ and $\widehat{R}_{t+1} = (\rho_k/R)[(1-\alpha)(\widehat{h}_{t+1} - \widehat{k}_{t+1}) - \widehat{\mu}_{t+1}]$. The resource constraint follows from differentiating $k_t^{\alpha} h_t^{1-\alpha} = c_t + k_{t+1} - (1-\delta)k_t$ and dividing by y^*.

13. A closed-form special case. Let $\delta = 1$ and $g = 0$, so that $y_t = k_t^{\alpha} h_t^{1-\alpha}$ and $k_{t+1} = y_t - c_t$.

(a) Guess that the household saves a fraction s_t of output, $k_{t+1} = s_t y_t$ and $c_t = (1 - s_t) y_t$, and show that the Euler equation becomes the linear recursion

$$q_t = 1 + \alpha\beta\, \mathbb{E}_t\Big[\frac{q_{t+1}}{\mu_{t+1}}\Big], \qquad q_t \equiv \frac{1}{1 - s_t}.$$

Deduce that with a constant markup $s = \alpha\beta/\mu$.

(b) Show that hours do not depend on the capital stock,

$$h_t^{1+\varphi} = \frac{1-\alpha}{\gamma\, \mu_t\, (1 - s_t)},$$

and that if $\widehat{\mu}_t$ follows an AR(1) with persistence ρ, then to first order

$$\widehat{h}_t = -\frac{1}{(1+\varphi)\,(1 - s\rho)}\, \widehat{\mu}_t, \qquad \widehat{y}_t = \alpha \widehat{k}_t + (1-\alpha)\widehat{h}_t.$$

(c) Interpret the factor $1/(1 - s\rho)$ and check the cases $\rho = 0$ and $\rho = 1$. Compute $\widehat{h}_t$ and $\widehat{y}_t$ on impact for $\alpha = 1/3$, $\beta = 0.95$, $\mu = 1.1$, $\varphi = 1$, $\rho = 0.95$ and $\widehat{\mu}_t = 1\%$.

Answer. (a) With $\delta = 1$ the Euler equation is $1/c_t = \beta \mathbb{E}_t[(1/c_{t+1})(\alpha/\mu_{t+1})y_{t+1}/k_{t+1}]$. Under the guess, $c_{t+1} = (1 - s_{t+1})y_{t+1}$ and $k_{t+1} = s_t y_t$, so

$$\frac{1}{(1-s_t)y_t} = \frac{\alpha\beta}{s_t y_t}\,\mathbb{E}_t\Big[\frac{1}{(1-s_{t+1})\mu_{t+1}}\Big] \quad \Longleftrightarrow \quad \frac{s_t}{1-s_t} = \alpha\beta\,\mathbb{E}_t\Big[\frac{q_{t+1}}{\mu_{t+1}}\Big].$$

Since $s_t/(1-s_t) = q_t - 1$, this is the stated recursion, which is *linear* in q – the guess is therefore exactly verified. With $\mu_t = \mu$ constant, $q = 1+\alpha\beta q/\mu$, i.e. $q = (1-\alpha\beta/\mu)^{-1}$ and $s = 1-1/q = \alpha\beta/\mu$. (b) The labor-supply condition is $\gamma c_t h_t^{\varphi} = (1/\mu_t)(1-\alpha)y_t/h_t$, i.e. $\gamma(1-s_t)y_t h_t^{1+\varphi} = (1-\alpha)y_t/\mu_t$: output cancels and the first expression follows. In logs,

$$(1+\varphi)\widehat{h}_t = -\widehat{\mu}_t - \widehat{(1-s_t)} = -\widehat{\mu}_t + \widehat{q}_t.$$

Log-linearizing the recursion, $\widehat{q}_t = s\,\mathbb{E}_t[\widehat{q}_{t+1} - \widehat{\mu}_{t+1}]$ (using $\alpha\beta/\mu = s$ and $q(1-s) = 1$). Guessing $\widehat{q}_t = -\Gamma\widehat{\mu}_t$ and using $\mathbb{E}_t\widehat{\mu}_{t+1} = \rho\widehat{\mu}_t$ gives $\Gamma = s\rho(1+\Gamma)$, so $\Gamma = s\rho/(1-s\rho)$ and

$$(1+\varphi)\widehat{h}_t = -\Big[1 + \frac{s\rho}{1-s\rho}\Big]\widehat{\mu}_t = -\frac{\widehat{\mu}_t}{1-s\rho}.$$

(c) The direct term, $-\widehat{\mu}_t/(1+\varphi)$, is the pure *labor-wedge* effect: a higher markup lowers the real wage relative to the marginal product of labor, and households move down their labor-supply curve with elasticity $1/(1+\varphi)$. The factor $1/(1-s\rho) \geq 1$ is the *capital-wedge* effect: a markup that is expected to persist also lowers the expected return on capital, so the saving rate falls, consumption today rises relative to output, the marginal rate of substitution rises further and hours fall by more – and the induced fall in capital feeds back into lower output tomorrow. Persistence therefore *amplifies* the contraction. At $\rho = 0$ the shock is purely transitory, $\widehat{q}_t = 0$ and only the labor wedge operates; at $\rho = 1$, $\widehat{s}_t = -\widehat{\mu}_t$ exactly, as $s = \alpha\beta/\mu$. Numbers: $s = \alpha\beta/\mu = (1/3)(0.95)/1.1 = 0.288$, $s\rho = 0.273$, so

$$\widehat{h}_t = -\frac{1}{2 \times 0.727}\,\widehat{\mu}_t = -0.69\,\widehat{\mu}_t, \qquad \widehat{y}_t = \tfrac{2}{3} \times (-0.69)\,\widehat{\mu}_t = -0.46\,\widehat{\mu}_t$$

on impact ($\widehat{k}_t = 0$): a 1% markup shock lowers hours by 0.69% and output by 0.46%. Without the capital channel ($\rho = 0$) the figures would be 0.50% and 0.33%.

14. (Optional, computational) Calibrate the model of Part C, feed in a markup process with $\rho = 0.95$ and σ_ε chosen to match the volatility of the measured labor wedge, simulate 50 years of quarterly data, HP-filter with $\lambda = 1600$ and report the variance-covariance matrix of $(C_t, H_t, K_t, X_t, Y_t, \mu_t)$. Which moments does the model match, and which does it miss?

Answer. The model produces the right *signs*: C, H, X and Y all move together, as in the data. Two counterfactual features stand out. (i) *Volatility*: given $\widehat{\mu}_t = -\widehat{\varepsilon}_t/(\varepsilon - 1)$ and the small coefficient computed in Question 15, enormous shocks to ε are needed to generate business-cycle-sized movements in hours. (ii) *Labor share*: the model ties the labor share one-for-one to the inverse markup, $s_{Lt} = (1-\alpha)/\mu_t$, so recessions caused by markup shocks have a *low* labor share. In US data the labor share is, if anything, mildly countercyclical at business-cycle frequencies, which is hard to reconcile with markup shocks as the dominant driver. Measured TFP is also constant by Question 5, whereas CKM find a large efficiency wedge. This is the usual verdict: markup shocks are a plausible *component* of the labor wedge, not its main source.

15. Discussion.

(a) An econometrician applies business cycle accounting to data from this economy. Which wedges does she find, and what would she conclude?

(b) In the New Keynesian model of Lecture 2, a shock to ε_t appears as the cost-push shock u_t in the Phillips curve $\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + u_t$. How is u_t related to the markup shock of this exercise? Why does the shock create a trade-off for the central bank in the NK model but not here?

Answer. (a) A labor wedge and an investment wedge moving together, with no efficiency wedge and no government-consumption wedge. She would conclude that the recession is a "labor market" recession and should look for models in which the gap between the marginal rate of substitution and the marginal product of labor widens – but she could *not* tell a markup shock from a labor tax, a union markup (Exercise 2), a search friction or a sticky wage. That is exactly CKM's methodological point: BCA narrows the class of candidate models, it does not identify the shock. (b) Here markups are *flexible* and the economy is always at its (distorted) natural allocation: the markup shock moves y_t^n itself, so the output gap $x_t = y_t - y_t^n$ stays zero and inflation is irrelevant. In the NK model prices are sticky, so a desired-markup shock cannot be passed into prices immediately; it moves the gap between the desired and the actual markup, which is precisely the cost-push term u_t. The central bank then faces a trade-off because stabilizing π_t requires moving x_t away from zero – the "divine coincidence" fails. In other words: the same primitive shock is a *supply* shock when prices are flexible and a *policy trade-off* when they are not.

Part II – The New Keynesian model

Objective. The four exercises of this part follow Lecture 2. Exercise 4 solves a two-period model in which the first-period price is fixed, so that output is determined by demand, and then measures the wedges of Part I in that economy. Exercise 5 derives the Phillips curve from the non-linear equilibrium conditions of the Calvo model. Exercise 6 compares the welfare gain from higher demand, which is of first order under monopolistic competition, with a firm's loss from not adjusting its price, which is of second order. Exercise 7 shows that the optimal monetary policy is then not time consistent.

Reminder: the three-equation model. With preferences $C^{1-\sigma}/(1-\sigma) - N^{1+\varphi}/(1+\varphi)$, a continuum of varieties with elasticity of substitution ε, technology $Y_t(i) = Z_t N_t(i)$ and Calvo pricing with reset probability $1-\theta$, the log-linearized equilibrium is

$$\begin{aligned} &\text{(DIS)} && x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\big(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n\big) \\ &\text{(NKPC)} && \pi_t = \beta\, \mathbb{E}_t \pi_{t+1} + \kappa\, x_t \ \big[+\ u_t\big] \\ &\text{(rule)} && i_t = \rho + \phi_\pi\, \pi_t + \phi_y\, x_t + v_t \end{aligned}$$

where $x_t = y_t - y_t^n$ is the *output gap* with respect to the natural (flexible-price) level of output, σ is the inverse elasticity of intertemporal substitution, φ the inverse Frisch elasticity, $\mu \equiv \varepsilon/(\varepsilon-1)$ the desired gross markup, mc_t real marginal cost, $\rho \equiv -\log\beta$, and

$$\big(Y_t^n\big)^{\sigma+\varphi} = \frac{Z_t^{1+\varphi}}{\mu}, \qquad \widehat{mc}_t = (\sigma+\varphi)\, x_t, \qquad \lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}, \qquad \kappa \equiv \lambda\,(\sigma+\varphi).$$

Exercises 5 and 6 derive these expressions. Exercise 7 takes them as given.

Exercise 4 – Sticky prices in a two-period model

The economy (Mankiw & Weinzierl, 2011; Moll, EC2B1 Lecture 7). There are two periods, $t = 1, 2$, and a preliminary period 0 in which a reference price P_0 is set.

- A representative household owns the firms and solves

$$\max_{C_1, C_2}\ U(C_1) + \beta\, U(C_2) \quad \text{s.t.} \quad P_1 C_1 + \frac{P_2 C_2}{1+i_1} = \Omega, \qquad U(C) = \frac{C^{1-\sigma}}{1-\sigma},$$

  where i_1 is the nominal interest rate and Ω the present value of nominal profits. As in Lecture 2, $1/\sigma$ is the elasticity of intertemporal substitution (EIS).

- Technology is linear in capital: $Y_1^S = A_1 K_1$ with K_1 given, $Y_2 = A_2 K_2$ and $K_2 = I_1$ (capital fully depreciates). Firms maximize $\Omega = P_1(Y_1 - I_1) + P_2 A_2 K_2/(1+i_1)$.

- Money demand follows the quantity theory, $M_t = P_t C_t$.

- Monetary policy sets (i_1, M_2), and M_1 adjusts to implement i_1.

- Inflation and the real rate: $1 + \pi_2 = P_2/P_1$ and $1 + r_1 = (1+i_1)P_1/P_2$.

Throughout define

$$\Psi \equiv A_2\,(\beta A_2)^{-1/\sigma}.$$

Part A – Flexible prices: monetary neutrality

1. Household and firm. Derive the Euler equation and show that it can be written $C_1 = C_2\big[\beta(1+r_1)\big]^{-1/\sigma}$. Show that the firm's problem has an interior solution only if

$$1 + r_1 = A_2,$$

and that in that case the real value of the firm is $W \equiv \Omega/P_1 = A_1 K_1$.

Answer. The household's first-order conditions are $U'(C_1) = \lambda P_1$ and $\beta U'(C_2) = \lambda P_2/(1+i_1)$, hence

$$\frac{U'(C_1)}{\beta\, U'(C_2)} = \frac{(1+i_1)P_1}{P_2} = 1 + r_1 \qquad \Longleftrightarrow \qquad C_1^{-\sigma} = \beta\,(1+r_1)\,C_2^{-\sigma},$$

which is the stated Euler equation. Dividing the firm's objective by P_1 and using $1 + r_1 = (1+i_1)P_1/P_2$, $W = \max_{I_1}\{A_1K_1 - I_1 + A_2 I_1/(1+r_1)\} = A_1K_1 + I_1\big[A_2/(1+r_1) - 1\big]$. The return to investment is constant, so the bracket must vanish in equilibrium. Indeed, were A_2 above $1+r_1$, the firm would demand an unbounded amount of capital; were it below, the firm would not invest at all and output in period 2 would be zero. Hence $1 + r_1 = A_2$ and $W = A_1K_1$: capital demand is infinitely elastic at that rate.

2. Real allocation. Combining the Euler equation, the budget constraint and goods-market clearing ($C_1 + I_1 = Y_1$, $C_2 = Y_2$), show that

$$C_1 = \frac{\Psi}{1+\Psi}A_1K_1, \quad I_1 = K_2 = \frac{1}{1+\Psi}A_1K_1, \quad C_2 = Y_2 = \frac{A_2}{1+\Psi}A_1K_1, \quad Y_1 = A_1K_1, \quad 1{+}r_1 = A_2.$$

Answer. With $1 + r_1 = A_2$, the Euler equation gives $C_1 = C_2(\beta A_2)^{-1/\sigma}$ and the budget constraint $C_1 + C_2/A_2 = A_1K_1$. Substituting $C_2 = A_2 I_1$ into the Euler equation, $C_1 = A_2 I_1(\beta A_2)^{-1/\sigma} = \Psi I_1$, and the period-1 resource constraint $C_1 + I_1 = A_1K_1$ gives $I_1(1+\Psi) = A_1K_1$. The rest follows.

3. Nominal variables and neutrality. Show that

$$P_1 = \frac{1+\Psi}{A_1K_1}\frac{M_2}{1+i_1}, \qquad P_2 = \frac{1+\Psi}{A_2A_1K_1}M_2, \qquad M_1 = \Psi\frac{M_2}{1+i_1}, \qquad 1+\pi_2 = \frac{1+i_1}{A_2}.$$

In what sense is money neutral? In what sense does the classical dichotomy hold?

Answer. From $P_2C_2 = M_2$ and $C_2 = A_2A_1K_1/(1+\Psi)$ we get P_2; from $1+r_1 = A_2 = (1+i_1)P_1/P_2$ we get $P_1 = A_2P_2/(1+i_1)$, which is the stated expression; and $M_1 = P_1C_1 = P_1\Psi A_1K_1/(1+\Psi) = \Psi M_2/(1+i_1)$. Finally $1 + \pi_2 = P_2/P_1 = (1+i_1)/A_2$, which is just the Fisher equation with $1 + r_1 = A_2$. *Neutrality*: the real block (Question 2) can be written entirely in real variables and does not involve $(i_1, M_1, M_2, P_1, P_2)$; the real allocation is identical to the one in the model without money. Policy affects only P_1, P_2 and inflation: a cut in i_1 or a rise in M_2 raises P_1 and P_2 proportionally. *Classical dichotomy*: the real and the nominal block can be solved *recursively* – first the real block, then the nominal one – not merely simultaneously.

Part B – Sticky prices: monetary non-neutrality

Assume now that P_1 is fixed at P_0: firms set it before they know A_1, A_2 and β. The price P_2 remains flexible. Output in period 1 is determined by the short side of the market,

$$Y_1 = \min\{Y_1^D, Y_1^S\}, \qquad Y_1^D = C_1 + I_1, \qquad Y_1^S = A_1K_1,$$

and parameters are such that $Y_1^D \le Y_1^S$.

4. The equilibrium allocation. Show that

$$C_1 = \Psi \frac{M_2}{(1+i_1)P_1}, \quad I_1 = \frac{M_2}{(1+i_1)P_1}, \quad Y_1 = (1+\Psi)\frac{M_2}{(1+i_1)P_1}, \quad C_2 = Y_2 = A_2\frac{M_2}{(1+i_1)P_1},$$

and $P_2 = (1+i_1)P_1/A_2$. [Hint: start from the firm's investment condition, then use the period-2 money market, then the Euler equation.]

Answer. The firm's investment FOC is unchanged – it involves P_1, P_2, i_1 and A_2 only – and gives $P_1 = P_2A_2/(1+i_1)$, i.e. $P_2 = (1+i_1)P_1/A_2$. The period-2 money market then gives $C_2 = M_2/P_2 = A_2M_2/[(1+i_1)P_1]$, and $I_1 = K_2 = C_2/A_2 = M_2/[(1+i_1)P_1]$. The Euler equation with $1+r_1 = A_2$ gives $C_1 = C_2(\beta A_2)^{-1/\sigma} = \Psi M_2/[(1+i_1)P_1]$, and $Y_1 = Y_1^D = C_1 + I_1$.

5. Non-neutrality and the output gap. Show that monetary policy now moves real variables, and that the output gap is

$$\frac{Y_1}{Y_1^S} = \frac{(1+\Psi)\,M_2}{(1+i_1)P_1A_1K_1} \leq 1.$$

What value of i_1 closes the gap? Interpret it.

Answer. Every real quantity in Question 4 is proportional to $M_2/[(1+i_1)P_1]$, which is a nominal object the central bank controls: a cut in i_1 or a rise in M_2 raises C_1, I_1, Y_1, C_2 and Y_2. Monetary neutrality fails. Setting $Y_1 = A_1K_1$ gives

$$1 + i_1^* = \frac{(1+\Psi)M_2}{P_1\,A_1K_1}.$$

This is the "natural" nominal rate in this model: at $i_1 = i_1^*$ the sticky-price allocation coincides with the flexible-price one of Part A (compare with the expression for P_1 in Question 3 – the same equation, read as determining i_1 given P_1 rather than P_1 given i_1). With flexible prices P_1 adjusts to satisfy it automatically; with P_1 fixed, the central bank must deliver it. If $i_1 > i_1^*$ demand is insufficient and output falls below potential.

6. Demand shocks. Consider two demand shocks: households expect lower future productivity ($A_2 \downarrow$), or they become more patient ($\beta \uparrow$).

(a) Show that $\beta \uparrow$ always lowers Y_1, and that $A_2 \downarrow$ lowers Y_1 if and only if the EIS is below one, $1/\sigma < 1$.

(b) Show that with flexible prices neither shock affects Y_1. Why does a fixed price change the answer?

Answer. (a) $\Psi = A_2(\beta A_2)^{-1/\sigma} = \beta^{-1/\sigma}A_2^{1-1/\sigma}$. Clearly $\partial\Psi/\partial\beta < 0$, so $\beta \uparrow$ lowers Ψ, hence C_1 and $Y_1 = (1+\Psi)M_2/[(1+i_1)P_1]$. And $\partial\Psi/\partial A_2 > 0$ iff $1 - 1/\sigma > 0$, i.e. $1/\sigma < 1$: with a low EIS, a lower expected return makes households want to consume *less* today (the income effect dominates the substitution effect), so $A_2 \downarrow$ reduces C_1 and causes a recession. With $1/\sigma > 1$ the substitution effect dominates and the shock would be expansionary. (b) With flexible prices $Y_1 = A_1K_1$ regardless: a change in Ψ reallocates period-1 output between C_1 and I_1 but cannot change the total, because output is supply-determined. With P_1 fixed, output is demand-determined, so anything that shifts desired spending shifts output one-for-one.

7. The paradox of thrift. Suppose $\beta \uparrow$. Compare the responses of C_1, I_1, Y_1, C_2 and Y_2 under flexible and sticky prices, and explain the difference. What is the role of the condition $\Omega = P_1 Y_1$?

Answer. *Flexible prices*: $C_1 \downarrow$, $I_1 \uparrow$, $C_2 = Y_2 \uparrow$, Y_1 unaffected. Desired saving translates one-for-one into investment and future output: thrift "works". *Sticky prices*: $C_1 \downarrow$, $Y_1 \downarrow$, while I_1, C_2 and Y_2 are *unaffected* – they depend only on $M_2/[(1+i_1)P_1]$. Everyone wants to save more and aggregate saving does not rise at all: the *paradox of thrift*. Mechanism: combining the household budget constraint with the firm's investment condition $P_1 = P_2 A_2/(1+i_1)$ gives $\Omega = P_1 Y_1$ with $Y_1 = Y_1^D$ – *one person's spending is someone else's income*. This closes a Keynesian-cross loop: $C_1 \downarrow \Rightarrow Y_1^D \downarrow \Rightarrow Y_1 \downarrow \Rightarrow \Omega \downarrow \Rightarrow C_1 \downarrow$ and so on. Here the loop runs through dividends; in richer models it also runs through labor income. Two forces act on I_1: the direct desire to save more pushes it up, the fall in income pushes it down, and in equilibrium they exactly cancel.

8. The real interest rate. Show that $1 + r_1 = A_2$ also holds when P_1 is fixed, so that a change in i_1 does not move the real rate. Which assumption is responsible? Through which channel does monetary policy affect output here? Compare with the dynamic IS curve $x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\big(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n\big)$ of the three-equation model.

Answer. From $P_2 = (1+i_1)P_1/A_2$ and $1 + r_1 = (1+i_1)P_1/P_2$ we get $1 + r_1 = A_2$ exactly as under flexible prices. The culprit is the *linear* technology $Y_2 = A_2 K_2$: capital demand is infinitely elastic at $1+r_1 = A_2$, so the real rate is pinned down by technology whatever policy does. The channel through which policy works here is instead the quantity-theory money demand: C_1 and I_1 are proportional to the real value of the end-of-period money stock discounted to period 1, $M_2/[(1+i_1)P_1]$. This is a weakness of the two-period model as an expositional device: in the three-equation NK model the transmission mechanism is precisely the one missing here – policy moves the real rate $i_t - \mathbb{E}_t \pi_{t+1}$ relative to the natural rate r_t^n, and the output gap responds with elasticity $1/\sigma$. With diminishing returns in period 2 ($Y_2 = A_2 K_2^\theta$, $\theta < 1$) capital demand slopes down and the pass-through reappears.

Part C – Back to business cycle accounting

9. Measured wedges in a demand recession. An econometrician observes the economy of Part B in a recession ($Y_1 < A_1 K_1$) and measures the wedges of the prototype economy of Part I, with $y_t = A_t k_t$, full depreciation and no labor.

(a) Show that the measured efficiency wedge is

$$A_1^{\text{meas}} = \frac{Y_1}{K_1} = \frac{(1+\Psi)M_2}{(1+i_1)P_1 K_1} < A_1.$$

(b) Show that the measured investment wedge is zero, $\tau_{x1} = 0$.

(c) What would she conclude? Compare with Question 9 of Exercise 2 and with CKM's finding that the efficiency wedge accounts for most of the fall in output in the Great Depression.

Answer. (a) The prototype's production function is $y_1 = A_1^{\text{meas}} k_1$ with $k_1 = K_1$ observed and $y_1 = Y_1$ observed, which gives the expression directly. True technology A_1 has not changed: the capital stock is fully employed in the accounts but output is lower, so the residual falls. (b) The prototype Euler equation with $\delta = 1$ and $F(k) = k$ reads $U'(C_1)(1+\tau_{x1}) = \beta U'(C_2) A_2^{\text{meas}}$, and $A_2^{\text{meas}} = A_2$ since period 2 is flexible. The household's Euler equation in the detailed economy is $C_1^{-\sigma} = \beta C_2^{-\sigma} A_2$, since $1 + r_1 = A_2$ (Question 8). Hence $\tau_{x1} = 0$: the *entire* recession is attributed to the efficiency wedge. (c) She would diagnose a pure technology recession, although the shock was a shortfall in demand (or a monetary policy that was too tight) and no technology changed. This is

the mirror image of Exercise 2: there, sticky *wages* with monetary shocks generated movements in the *labor* wedge only and could not produce the observed fall in measured TFP; here, sticky *prices* with demand-determined output generate movements in the *efficiency* wedge, because inputs are measured as fully employed while output is not. This is the standard "capacity utilization / labor hoarding" reading of the Solow residual, and it means that CKM's finding of a dominant efficiency wedge does *not* by itself rule out demand-driven recessions: it rules out models in which insufficient demand shows up only as a labor wedge.

10. Partial stickiness. Suppose instead that P_1 is partially flexible, $P_1 = P_0^{1-\omega}\tilde{P}_1^{\omega}$ with $\omega \in [0,1]$, where $\tilde{P}_1$ is the flexible-price level from Question 3. How does the output gap of Question 5 depend on ω? What do the cases $\omega = 1$ and $\omega = 0$ correspond to?

Answer. Write the gap in logs: $\log(Y_1/Y_1^S) = \log[(1+\Psi)M_2/((1+i_1)A_1K_1)] - \log P_1 = \log\tilde{P}_1 - \log P_1$, using Question 3. With partial stickiness $\log\tilde{P}_1 - \log P_1 = (1-\omega)(\log\tilde{P}_1 - \log P_0)$: the output gap is the price gap, scaled down by $1-\omega$. At $\omega = 1$ prices are fully flexible, $P_1 = \tilde{P}_1$, and the gap is zero (Part A); at $\omega = 0$ prices are fully rigid and the gap is the entire price gap (Part B). Intermediate ω gives the realistic case: a demand shock is absorbed partly by prices and partly by quantities, which is what the Phillips curve of the three-equation model describes: a flatter Phillips curve (a smaller κ) corresponds to more price stickiness (a smaller ω), and the limit $\kappa \to 0$ to the fully demand-determined case of Part B.

Exercise 5 – From the Calvo system to the New Keynesian Phillips curve

Objective. The Calvo model is usually presented in log-linear form. This exercise starts from its non-linear equilibrium conditions, reduces them to three equations, finds the zero-inflation steady state, and log-linearizes last. The derivation shows where $\lambda = (1-\theta)(1-\beta\theta)/\theta$ comes from and why price dispersion can be ignored to first order.

The system. Labor is the only input, $y_{jt} = z_t h_{jt}$, utility is $U(C,H) = \ln C - v(H)$ with $v' > 0$, $v'' > 0$, varieties have elasticity of substitution $\varepsilon > 1$ and a firm may reset its price each period with probability $1-\theta$. Writing P_t for the ideal (consumption) price index, $\tilde{P}_t$ for the "production" price index, m_t for real marginal cost summed over the expected life of a price and d_t for the corresponding discount factor, an equilibrium is a collection of processes for $C_t(s^t), H_t(s^t), W_t(s^t), Q_0^t(s^t), P_t(s^t), \tilde{P}_t(s^t), m_t(s^t), d_t(s^t)$

such that

$$C_t\, v'(H_t) = \frac{W_t}{P_t} \tag{1}$$

$$C_t = z_t H_t \Big(\frac{\tilde{P}_t}{P_t}\Big)^{\varepsilon} \tag{2}$$

$$P_{t+1}^{1-\varepsilon} = \theta\, P_t^{1-\varepsilon} + (1-\theta)\Big(\frac{\varepsilon\, m_{t+1} P_{t+1}}{(\varepsilon-1) d_{t+1}}\Big)^{1-\varepsilon} \tag{3}$$

$$\frac{1}{C_t} = \beta\, \frac{\Pi_0^{t+1}(s^{t+1})\, Q_0^t(s^t)\, P_t(s^t)}{\Pi_0^t(s^t)\, Q_0^{t+1}(s^{t+1})\, P_{t+1}(s^{t+1})\, C_{t+1}(s^{t+1})} \tag{4}$$

$$\tilde{P}_{t+1}^{-\varepsilon} = \theta\, \tilde{P}_t^{-\varepsilon} + (1-\theta)\Big(\frac{\varepsilon\, m_{t+1} P_{t+1}}{(\varepsilon-1) d_{t+1}}\Big)^{-\varepsilon} \tag{5}$$

$$m_t = \frac{W_t}{z_t P_t} + \theta \sum_{s^{t+1} \succ s^t} \frac{Q_0^{t+1}\, C_{t+1}\, P_{t+1}^{\varepsilon+1}}{Q_0^t\, C_t\, P_t^{\varepsilon+1}}\, m_{t+1} \tag{6}$$

$$d_t = 1 + \theta \sum_{s^{t+1} \succ s^t} \frac{Q_0^{t+1}\, C_{t+1}\, P_{t+1}^{\varepsilon}}{Q_0^t\, C_t\, P_t^{\varepsilon}}\, d_{t+1} \tag{7}$$

The reset price is $P_t^* = \dfrac{\varepsilon}{\varepsilon-1}\dfrac{m_t P_t}{d_t} = \mu \dfrac{m_t P_t}{d_t}$ with $\mu \equiv \varepsilon/(\varepsilon-1)$, a markup over a weighted average of current and expected future marginal costs.

Part A – Reducing the system

Define $\pi_t \equiv P_t/P_{t-1}$ and $\chi_t \equiv \tilde{P}_t/P_t$.

1. Eliminating the intertemporal price. Use (4) to show that (6) and (7) become

$$m_t = \frac{W_t}{z_t P_t} + \theta\beta\, \mathbb{E}_t\big[\pi_{t+1}^{\varepsilon}\, m_{t+1}\big], \qquad d_t = 1 + \theta\beta\, \mathbb{E}_t\big[\pi_{t+1}^{\varepsilon-1}\, d_{t+1}\big].$$

Answer. Rearranging (4), $\dfrac{Q_0^{t+1}}{Q_0^t} = \beta \dfrac{\Pi_0^{t+1}}{\Pi_0^t}\dfrac{P_t}{P_{t+1}}\dfrac{C_t}{C_{t+1}}$. Hence

$$\frac{Q_0^{t+1} C_{t+1} P_{t+1}^{\varepsilon+1}}{Q_0^t C_t P_t^{\varepsilon+1}} = \beta \frac{\Pi_0^{t+1}}{\Pi_0^t}\frac{P_t}{P_{t+1}}\frac{C_t}{C_{t+1}}\frac{C_{t+1}}{C_t}\Big(\frac{P_{t+1}}{P_t}\Big)^{\varepsilon+1} = \beta \frac{\Pi_0^{t+1}}{\Pi_0^t}\, \pi_{t+1}^{\varepsilon},$$

and the sum over $s^{t+1} \succ s^t$ of $(\Pi_0^{t+1}/\Pi_0^t) \times (\cdot)$ is the conditional expectation $\mathbb{E}_t[\cdot]$. The same computation with exponent ε instead of $\varepsilon+1$ gives $\pi_{t+1}^{\varepsilon-1}$ in the equation for d_t. Note how the consumption terms cancel: with log utility the nominal SDF is $\beta P_t/(P_{t+1}C_{t+1}/C_t)$ and the C_{t+1} in the numerator of (6) is exactly what the SDF discounts.

2. An explicit solution for d_t. Divide (3) by $P_t^{1-\varepsilon}$ and solve for d to show that

$$d_t = \Big[\frac{1-\theta}{\pi_t^{1-\varepsilon} - \theta}\Big]^{\frac{1}{1-\varepsilon}} \frac{\varepsilon\, m_t\, \pi_t}{\varepsilon - 1}, \qquad \text{equivalently} \qquad \frac{P_t^*}{P_t} = \frac{1}{\pi_t}\Big[\frac{\pi_t^{1-\varepsilon} - \theta}{1-\theta}\Big]^{\frac{1}{1-\varepsilon}}.$$

Interpret the second expression. What is P_t^*/P_t when $\pi_t = 1$?

Answer. Dividing (3) by $P_t^{1-\varepsilon}$ gives $\pi_{t+1}^{1-\varepsilon} = \theta + (1-\theta)\big(\varepsilon m_{t+1}\pi_{t+1}/[(\varepsilon-1)d_{t+1}]\big)^{1-\varepsilon}$, hence $\mu m_{t+1}\pi_{t+1}/d_{t+1} = \big[(\pi_{t+1}^{1-\varepsilon}-\theta)/(1-\theta)\big]^{1/(1-\varepsilon)}$, and shifting the time index back by one period gives the stated formula. Since $P_t^*/P_t = \mu m_t/d_t$, dividing by π_t gives the second expression. The second expression is the price index identity read backwards: the index P_t is a CES average of the fraction θ of prices inherited from $t-1$ and the fraction $1-\theta$ reset at t. If prices are rising, the inherited prices are too *low* relative to the index, so the resetters must price *above* the index: $P_t^*/P_t > 1$ iff $\pi_t > 1$. At $\pi_t = 1$ the bracket equals 1 and $P_t^* = P_t$: nobody who resets wants to move away from the index.

3. A relation between m and π. Substitute Question 2 into the recursion for d_t to obtain

$$\Big[\frac{1-\theta}{\pi_t^{1-\varepsilon}-\theta}\Big]^{\frac{1}{1-\varepsilon}}\frac{\varepsilon\, m_t\,\pi_t}{\varepsilon-1} = 1+\theta\beta\,\mathbb{E}_t\left[\pi_{t+1}^{\varepsilon}\Big[\frac{1-\theta}{\pi_{t+1}^{1-\varepsilon}-\theta}\Big]^{\frac{1}{1-\varepsilon}}\frac{\varepsilon\, m_{t+1}}{\varepsilon-1}\right].$$

Answer. Immediate: evaluate the expression of Question 2 at $t+1$, note that $\pi_{t+1}^{\varepsilon-1}d_{t+1}$ contains one power of π_{t+1} from d_{t+1} itself, so $\pi_{t+1}^{\varepsilon-1}d_{t+1} = \pi_{t+1}^{\varepsilon}[\cdot]^{1/(1-\varepsilon)}\mu m_{t+1}$, and substitute into $d_t = 1+\theta\beta\mathbb{E}_t[\pi_{t+1}^{\varepsilon-1}d_{t+1}]$.

4. A relation between χ and π. Divide (5) by $P_{t+1}^{-\varepsilon}$ and use Question 2 to show that

$$\chi_{t+1}^{-\varepsilon} = \theta\Big(\frac{\chi_t}{\pi_{t+1}}\Big)^{-\varepsilon} + (1-\theta)\left(\Big[\frac{1-\theta}{\pi_{t+1}^{1-\varepsilon}-\theta}\Big]^{\frac{1}{1-\varepsilon}}\pi_{t+1}\right)^{\varepsilon}.$$

This equation relates χ and π only.

Answer. Dividing (5) by $P_{t+1}^{-\varepsilon}$, $(\tilde{P}_{t+1}/P_{t+1})^{-\varepsilon} = \theta(\tilde{P}_t/P_t\cdot P_t/P_{t+1})^{-\varepsilon} + (1-\theta)(\mu m_{t+1}/d_{t+1})^{-\varepsilon}$, i.e. $\chi_{t+1}^{-\varepsilon} = \theta(\chi_t/\pi_{t+1})^{-\varepsilon} + (1-\theta)(P_{t+1}^*/P_{t+1})^{-\varepsilon}$. Substituting P_{t+1}^*/P_{t+1} from Question 2 gives the result: m and d enter only through the reset price, which Question 2 expresses as a function of inflation alone.

5. A relation between m, H and χ. Eliminate C_t between (1) and (2), then eliminate the real wage in the equation for m_t from Question 1, to get

$$m_t = H_t\,\chi_t^{\varepsilon}\,v'(H_t) + \theta\beta\,\mathbb{E}_t\big[\pi_{t+1}^{\varepsilon}\,m_{t+1}\big].$$

Why does z_t not appear?

Answer. From (2), $C_t = z_t H_t \chi_t^{\varepsilon}$; substituting into (1), $W_t/P_t = z_t H_t \chi_t^{\varepsilon} v'(H_t)$. Dividing by z_t gives the real marginal cost of the period, $W_t/(z_t P_t) = H_t\chi_t^{\varepsilon}v'(H_t)$, and substituting into the recursion of Question 1 gives the result. Productivity cancels because it scales the real wage and the marginal product of labor identically: with log utility, a productivity shock raises consumption and the real wage in exactly the proportion that leaves marginal cost unchanged. This is why a pure technology shock creates no inflation here – a version of the "divine coincidence".

Part B – Steady state and log-linearization

The system is now three equations (Questions 3–5) in $(m_t, \pi_t, \chi_t, H_t)$.

6. The zero-inflation steady state. Look for a steady state with $\pi_t = 1$ for all t and show that

$$\chi = 1, \qquad m = \frac{1}{\mu\,(1-\theta\beta)} = \frac{\varepsilon - 1}{\varepsilon(1-\theta\beta)}, \qquad H\,v'(H) = \frac{1}{\mu} = \frac{\varepsilon - 1}{\varepsilon}.$$

Interpret the last condition, and compare it with Exercise 2 (Question 4) and with the natural level of output of Lecture 2.

Answer. With $\pi = 1$ the bracket $[(1-\theta)/(\pi^{1-\varepsilon} - \theta)]^{1/(1-\varepsilon)}$ equals one. The equation of Question 4 becomes $\chi^{-\varepsilon} = \theta\chi^{-\varepsilon} + (1-\theta)$, i.e. $\chi^{-\varepsilon}(1-\theta) = 1-\theta$, so $\chi = 1$: with zero inflation there is no price dispersion and the two indices coincide. The equation of Question 3 becomes $\mu m = 1 + \theta\beta\mu m$, i.e. $\mu m(1-\theta\beta) = 1$. Substituting $\chi = 1$ into Question 5, $m = Hv'(H) + \theta\beta m$, so $Hv'(H) = m(1-\theta\beta) = 1/\mu$. Since $z = 1$ the marginal product of labor is 1 and $Hv'(H)$ is the marginal rate of substitution (recall $-U_H/U_C = Cv'(H)$ and $C = H$ at $\chi = 1$). The steady state therefore satisfies MRS/MPL $= 1/\mu < 1$: exactly the condition MRS/MPL $= v$ of Exercise 2, with the union's monopoly power replaced by the firms' market power, and exactly the natural output condition $(Y^n)^{\sigma+\varphi} = Z^{1+\varphi}/\mu$ of Lecture 2 specialized to $\sigma = 1$, $z = 1$. Market power acts like a constant tax on labor of $1 - 1/\mu = 1/\varepsilon$.

7. Price dispersion is transitory. Log-linearize the equation of Question 4 around $(\pi, \chi) = (1, 1)$ and show that

$$\widehat{\chi}_{t+1} = \theta\,\widehat{\chi}_t.$$

Conclude that $\widehat{\chi}_t \to 0$ monotonically from any initial condition, and explain why this justifies setting $\widehat{\chi}_t = 0$ in what follows. [Hint: show first that the function $\Lambda(\pi) \equiv \left[\frac{1-\theta}{\pi^{1-\varepsilon}-\theta}\right]^{\frac{1}{1-\varepsilon}}\pi$ satisfies $\Lambda(1) = 1$ and $d\log\Lambda/d\log\pi = -\theta/(1-\theta)$ at $\pi = 1$; note that $P_t^*/P_t = 1/\Lambda(\pi_t)$.]

Answer. Write $\Lambda(\pi) \equiv \left[\frac{1-\theta}{\pi^{1-\varepsilon}-\theta}\right]^{\frac{1}{1-\varepsilon}}\pi$, so $\Lambda(1) = 1$ and

$$\frac{d\log\Lambda}{d\log\pi} = -\frac{1}{1-\varepsilon}\frac{(1-\varepsilon)\pi^{1-\varepsilon}}{\pi^{1-\varepsilon}-\theta} + 1 = 1 - \frac{\pi^{1-\varepsilon}}{\pi^{1-\varepsilon}-\theta} \quad \xrightarrow{\pi\to 1} \quad 1 - \frac{1}{1-\theta} = -\frac{\theta}{1-\theta}.$$

The equation of Question 4 is $\chi_{t+1}^{-\varepsilon} = \theta(\chi_t/\pi_{t+1})^{-\varepsilon} + (1-\theta)\Lambda(\pi_{t+1})^{\varepsilon}$. Differentiating around the steady state,

$$-\varepsilon\widehat{\chi}_{t+1} = \theta\cdot(-\varepsilon)(\widehat{\chi}_t - \widehat{\pi}_{t+1}) + (1-\theta)\cdot\varepsilon\cdot\Big(-\frac{\theta}{1-\theta}\Big)\widehat{\pi}_{t+1} = -\varepsilon\theta\widehat{\chi}_t + \varepsilon\theta\widehat{\pi}_{t+1} - \varepsilon\theta\widehat{\pi}_{t+1},$$

so the inflation terms cancel exactly and $\widehat{\chi}_{t+1} = \theta\widehat{\chi}_t$. This is a *backward*-looking difference equation with no forcing term and coefficient $\theta \in (0,1)$: $\widehat{\chi}_t = \theta^t\widehat{\chi}_0 \to 0$, monotonically since $\theta > 0$. To first order, relative price dispersion is a pure initial condition: it is not driven by shocks and it dies out. So if the economy starts from the zero-inflation steady state, $\widehat{\chi}_t = 0$ for all t. (This is the familiar statement that price dispersion is second order in the Calvo model and can be ignored in a first-order approximation – but it matters for *welfare*, which is itself second order: see Exercise 6.)

8. The Phillips curve. Log-linearize the equations of Questions 3 and 5 around the zero-inflation steady state, set $\widehat{\chi}_t = 0$, and show that $\widehat{m}_t$ drops out, leaving

$$\boxed{\widehat{\pi}_t = \beta\,\mathbb{E}_t\widehat{\pi}_{t+1} + \lambda\,(1+\omega)\,\widehat{H}_t, \qquad \lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}, \qquad \omega \equiv \frac{H\,v''(H)}{v'(H)}.}$$

Answer. *Question 5.* Write $me^{\widehat{m}_t} = He^{\widehat{H}_t}\chi_t^{\varepsilon}v'(He^{\widehat{H}_t}) + \theta\beta m\,\mathbb{E}_t[e^{\varepsilon\widehat{\pi}_{t+1}+\widehat{m}_{t+1}}]$ and differentiate at $\widehat{\ }=0$:

$$m\widehat{m}_t = Hv'(H)\big[(1+\omega)\widehat{H}_t + \varepsilon\widehat{\chi}_t\big] + \theta\beta m\big[\varepsilon\mathbb{E}_t\widehat{\pi}_{t+1} + \mathbb{E}_t\widehat{m}_{t+1}\big],$$

using $\frac{d}{d\widehat{H}}\big[He^{\widehat{H}}v'(He^{\widehat{H}})\big] = Hv'(H)+H^2v''(H) = Hv'(H)(1+\omega)$. Dividing by m and using $Hv'(H) = m(1-\theta\beta)$ and $\widehat{\chi}_t = 0$:

$$\widehat{m}_t - \theta\beta\,\mathbb{E}_t\widehat{m}_{t+1} = (1-\theta\beta)(1+\omega)\widehat{H}_t + \theta\beta\varepsilon\,\mathbb{E}_t\widehat{\pi}_{t+1}. \qquad \text{(A)}$$

Question 3. With $\Lambda(\pi)$ as in Question 7, that equation reads

$$\Lambda(\pi_t)\,\mu\,m_t = 1 + \theta\beta\,\mathbb{E}_t\big[\pi_{t+1}^{\varepsilon-1}\,\Lambda(\pi_{t+1})\,\mu\,m_{t+1}\big],$$

and the steady state satisfies $\mu\,m\,(1-\theta\beta) = 1$. Differentiating and using $\widehat{\Lambda}_t = -\frac{\theta}{1-\theta}\widehat{\pi}_t$:

$$\mu m\Big[\widehat{m}_t - \frac{\theta}{1-\theta}\widehat{\pi}_t\Big] = \theta\beta\mu m\,\mathbb{E}_t\Big[\widehat{m}_{t+1} + \Big(\varepsilon - 1 - \frac{\theta}{1-\theta}\Big)\widehat{\pi}_{t+1}\Big],$$

and since $1 + \frac{\theta}{1-\theta} = \frac{1}{1-\theta}$,

$$\widehat{m}_t - \theta\beta\,\mathbb{E}_t\widehat{m}_{t+1} = \frac{\theta}{1-\theta}\widehat{\pi}_t + \theta\beta\Big(\varepsilon - \frac{1}{1-\theta}\Big)\mathbb{E}_t\widehat{\pi}_{t+1}. \qquad \text{(B)}$$

Combining. The left-hand sides of (A) and (B) are identical, so

$$(1-\theta\beta)(1+\omega)\widehat{H}_t + \theta\beta\varepsilon\,\mathbb{E}_t\widehat{\pi}_{t+1} = \frac{\theta}{1-\theta}\widehat{\pi}_t + \theta\beta\varepsilon\,\mathbb{E}_t\widehat{\pi}_{t+1} - \frac{\theta\beta}{1-\theta}\mathbb{E}_t\widehat{\pi}_{t+1}.$$

The terms in ε cancel – this is why the elasticity of substitution does not appear in the slope of the Phillips curve. Rearranging and multiplying by $(1-\theta)/\theta$ gives the boxed equation.

9. Mapping to the lecture. Take $v(H) = \gamma H^{1+\varphi}/(1+\varphi)$.

(a) Show that $\omega = \varphi$ and that the slope is $\lambda(1+\varphi)$.

(b) Suppose z_t is constant. Show that $\widehat{H}_t$ equals the output gap $x_t = y_t - y_t^n$ of Lecture 2, so that the boxed equation is $\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \kappa\,x_t$ with $\kappa = \lambda(\sigma+\varphi)$ and $\sigma = 1$.

(c) What changes if z_t is stochastic?

Answer. (a) $v'(H) = \gamma H^{\varphi}$ and $v''(H) = \gamma\varphi H^{\varphi-1}$, so $\omega = Hv''/v' = \varphi$. (b) With z constant and $\widehat{\chi}_t = 0$ we have $C_t = zH_t$, so $\widehat{C}_t = \widehat{Y}_t = \widehat{H}_t$. The natural level of hours solves $Hv'(H) = 1/\mu$ (Question 6), which does not move: it is a constant. Hence $\widehat{H}_t$ measures hours (and output) relative to their natural level, i.e. exactly the output gap. With $U = \ln C - v(H)$ we have $\sigma = 1$, so $\lambda(1+\varphi) = \lambda(\sigma+\varphi) = \kappa$. The real-marginal-cost form of the Phillips curve, $\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \lambda\widehat{mc}_t$, is recovered by noting $\widehat{mc}_t = (1+\varphi)\widehat{H}_t = (\sigma+\varphi)x_t$. (c) With stochastic z_t, $\widehat{Y}_t = \widehat{z}_t + \widehat{H}_t$ and natural output is $\widehat{y}_t^n = \frac{1+\varphi}{\sigma+\varphi}\widehat{z}_t$, so $\widehat{H}_t = \widehat{y}_t - \widehat{z}_t$ is no longer the gap. One has to re-express the right-hand side in terms of $x_t = \widehat{y}_t - \widehat{y}_t^n$: the Phillips curve is unchanged but its forcing variable is the output *gap*, not hours. As a check, in the special case $\sigma = 1$: $\widehat{y}_t^n = \widehat{z}_t$ and $\widehat{H}_t = \widehat{y}_t - \widehat{z}_t = x_t$ again – with log utility and linear technology, hours *are* the gap even with technology shocks.

10. Discussion. The parameter θ enters λ twice: through the frequency of price changes, $1-\theta$, and through the weight $\beta\theta$ that a firm puts on next period's marginal cost when it sets its price.

(a) Show that $\lambda \to \infty$ as $\theta \to 0$ and $\lambda \to 0$ as $\theta \to 1$, and interpret.

(b) For $\beta = 0.99$ and $\theta = 2/3$ (prices reset every three quarters on average), compute λ and κ for $\varphi = 1$. Compare with the estimates discussed in Lecture 2 ($\kappa \approx 0.006$–0.03).

(c) Which parameters could reconcile the two? Would a different ε help (see Question 8)?

Answer. (a) $\lambda = (1-\theta)(1-\beta\theta)/\theta$. As $\theta \to 0$ all prices reset every period and $\lambda \to \infty$: inflation reacts without limit to marginal cost – the flexible-price limit, in which the Phillips curve is vertical. As $\theta \to 1$ prices never change and $\lambda \to 0$: the Phillips curve is flat and marginal cost has no effect on inflation. (b) $\lambda = (1/3)(1-0.66)/(2/3) = (1/3)(0.34)(3/2) = 0.17$, and $\kappa = \lambda(\sigma+\varphi) = 0.17 \times 2 = 0.34$. This is one to two *orders of magnitude* above the empirical estimates – the "flat Phillips curve" puzzle of Lecture 2. (c) Two routes. (i) Longer price durations: matching $\kappa = 0.03$ with $\sigma + \varphi = 2$ requires $\lambda = 0.015$, i.e. $\theta \approx 0.89$, which means prices reset every nine quarters – far more rigid than the micro evidence. (ii) A weaker link from the output gap to marginal cost, i.e. a smaller $\sigma + \varphi$: this is the "real rigidity" channel, and the decomposition $\kappa = \lambda \times \Omega$ with Ω the elasticity of marginal cost to the gap (Gagliardone et al., 2025) says that the second route is the empirically relevant one. Changing ε does *not* help: Question 8 showed that all the ε terms cancel from the slope. The elasticity of substitution governs the *level* of the markup, and hence the welfare weights of Exercise 6, but not the slope of the Phillips curve.

Exercise 6 – Monopolistic competition and the aggregate demand externality

Objective. This exercise proves two results of Blanchard & Kiyotaki (1987) in the New Keynesian model of Lecture 2. First, under monopolistic competition an increase in aggregate demand raises welfare at first order, while a firm that does not adjust its price loses only at second order. Small costs of changing prices can then have large real effects. Second, no firm has a private incentive to bring this increase about, which is why it is called an aggregate demand *externality*.

The environment (Lecture 2). A representative household has preferences

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right], \qquad \text{so} \qquad \frac{W_t}{P_t} = N_t^{\varphi} C_t^{\sigma},$$

with σ the inverse EIS and φ the inverse Frisch elasticity. Firm $i \in [0,1]$ produces $Y_t(i) = Z_t N_t(i)$ and faces demand $Y_t(i) = \big(P_t(i)/P_t\big)^{-\varepsilon} Y_t$ with $\varepsilon > 1$. Real marginal cost is $mc_t \equiv W_t/(P_t Z_t)$, common across firms, and the desired gross markup is $\mu \equiv \varepsilon/(\varepsilon - 1)$. In equilibrium $C_t = Y_t = Z_t N_t$ (ignoring price dispersion, which is of second order by Exercise 5).

Part A – Natural output is inefficiently low

1. The flexible-price allocation. Show that with flexible prices every firm sets $P_t(i) = \mu\, W_t/Z_t$, so that $mc_t = 1/\mu$, and deduce the *natural* level of output

$$\big(Y_t^n\big)^{\sigma+\varphi} = \frac{Z_t^{1+\varphi}}{\mu}.$$

Answer. The firm maximizes $\big(P(i) - W_t/Z_t\big)\big(P(i)/P_t\big)^{-\varepsilon} Y_t$; the FOC gives $P(i) = \frac{\varepsilon}{\varepsilon-1}\frac{W_t}{Z_t} = \mu \frac{W_t}{Z_t}$ (see Exercise 3, Question 3). By symmetry $P_t(i) = P_t$, so $mc_t = W_t/(P_t Z_t) = 1/\mu$. Substituting the labor supply condition $W_t/P_t = N_t^{\varphi} C_t^{\sigma}$ and then $N_t = Y_t/Z_t$, $C_t = Y_t$,

$$\frac{1}{\mu} = \frac{N_t^{\varphi} C_t^{\sigma}}{Z_t} = \frac{(Y_t/Z_t)^{\varphi} Y_t^{\sigma}}{Z_t} = \frac{Y_t^{\sigma+\varphi}}{Z_t^{1+\varphi}}.$$

2. The efficient allocation. Show that a planner who maximizes period utility subject to $Y = ZN$, $C = Y$ equates $-U_N/U_C = Z_t$ and obtains $\left(Y_t^e\right)^{\sigma+\varphi} = Z_t^{1+\varphi}$. Deduce

$$\frac{Y_t^n}{Y_t^e} = \mu^{-\frac{1}{\sigma+\varphi}} < 1, \qquad \text{i.e.} \qquad y_t^n - y_t^e = -\frac{\log \mu}{\sigma+\varphi},$$

and compute the gap for $\varepsilon = 6$, $\sigma = \varphi = 1$.

Answer. The planner maximizes $C^{1-\sigma}/(1-\sigma) - N^{1+\varphi}/(1+\varphi)$ subject to $C = ZN$: the FOC is $C^{-\sigma}Z = N^{\varphi}$, i.e. $N^{\varphi}C^{\sigma} = Z$ and $Y^{\sigma+\varphi} = Z^{1+\varphi}$. Dividing by the natural allocation of Question 1 gives $(Y^n/Y^e)^{\sigma+\varphi} = 1/\mu$. With $\varepsilon = 6$, $\mu = 1.2$ and $\sigma + \varphi = 2$: $y^n - y^e = -\log(1.2)/2 = -9.1\%$. Monopolistic competition is *not* a small distortion. *But* note: this is a *level* distortion, present in every period and every state. It does not by itself make output demand-determined – with flexible prices the classical dichotomy still holds (Exercise 4, Part A) and the path of r_t or of M_t/P_t has no effect on real allocations.

Part B – The first-order welfare gain of a demand expansion

With sticky prices, output is determined by demand. Since $P_t(i) > W_t/Z_t$, firms are willing to serve extra demand at the posted price.

3. Welfare effect of a demand expansion. Consider raising output by dY_t at given prices. Show that

$$dU_t = C_t^{-\sigma}\left(1 - mc_t\right) dY_t, \qquad \text{equivalently} \qquad \frac{dU_t}{U_{Ct}\, C_t} = \left(1 - mc_t\right) x_t,$$

where $x_t \equiv y_t - y_t^n$ is the output gap in logs. Evaluate at $mc_t = 1/\mu$ and show that the coefficient on x_t is

$$\Phi \equiv 1 - \frac{1}{\mu} = \frac{1}{\varepsilon}.$$

Why is this gain of first order, when it would be of second order under perfect competition?

Answer. Totally differentiating period utility, $dU_t = U_{Ct}dC_t + U_{Nt}dN_t = C_t^{-\sigma}dY_t - N_t^{\varphi}dN_t$ (using $C = Y$). From $Y = ZN$, $dN_t = dY_t/Z_t$, so

$$dU_t = C_t^{-\sigma}dY_t - \frac{N_t^{\varphi}}{Z_t}dY_t = C_t^{-\sigma}dY_t\left[1 - \frac{N_t^{\varphi}C_t^{\sigma}}{Z_t}\right] = C_t^{-\sigma}\big(1 - mc_t\big)dY_t,$$

since $mc_t = W_t/(P_tZ_t) = N_t^{\varphi}C_t^{\sigma}/Z_t$ by the labor supply condition. Writing $dY_t = Y_t x_t$ for a small log deviation and $U_{Ct}C_t = C_t^{1-\sigma}$ gives the normalized version. At the natural allocation $mc = 1/\mu$, so $1 - mc = 1 - 1/\mu = 1 - (\varepsilon-1)/\varepsilon = 1/\varepsilon$. *Economics.* The social value of an extra unit is its price (what the household is willing to pay), the social cost is its marginal cost (the disutility of the labor it takes). The markup is a wedge between the two *on every unit*, so the gain from one more unit does *not* vanish at the margin: welfare is *linear* in x_t near the natural allocation. This is the exact opposite of the competitive case, where $mc = 1$, the bracket is zero, and the gain from extra output is second order. *Remark:* the lecture slide writes this expression as $C^{-\sigma-1}(1-1/\mu)x_t$; the correct normalization is $C^{1-\sigma}(1-1/\mu)x_t$, or equivalently $dU_t/(U_{Ct}C_t) = x_t/\varepsilon$.

4. The externality. All firms and the household would gain if every firm cut its price. Why does no firm do so on its own? Which two assumptions does the argument rely on?

Answer. Firm i takes the aggregate price index P_t and aggregate demand Y_t as given – correctly, since it is of measure zero. Cutting its own price raises $Y_t(i)$ by stealing market share from the others,

which is privately optimal only up to the point where $P_t(i) = \mu W_t/Z_t$; it has *no* effect on P_t and therefore none on aggregate demand Y_t. A coordinated cut by *all* firms (Blanchard–Kiyotaki) or an expansion engineered by the central bank through $r_t \downarrow$ (the NK version) shifts *every* firm's demand curve out, which is where the first-order gain of Question 3 comes from. No individual price setter can capture more than an infinitesimal share of that gain, so nobody starts it. The two assumptions doing the work are (i) a continuum of atomistic price setters, so that each one's price has a negligible effect on the index, and (ii) demand curves that shift with aggregate activity. Monopoly power *per se* is not enough: a single monopolist producing the whole output would internalize the effect.

5. The second-order private loss. Consider a firm that faces real marginal cost mc_t and aggregate demand Y_t and charges the relative price $p \equiv P_t(i)/P_t$, so that its real profit is $\Pi(p) = (p - mc_t)\, p^{-\varepsilon} Y_t$.

(a) Show that $\Pi'(p^*) = 0$ at $p^* = \mu\, mc_t$, that $\Pi(p^*) = (p^*)^{1-\varepsilon} Y_t/\varepsilon$, and that $\Pi''(p^*) = -(\varepsilon - 1)(p^*)^{-\varepsilon-1} Y_t$.

(b) Deduce that a firm charging $p = p^*(1+\xi)$ loses, as a fraction of its revenue,

$$\frac{\Pi(p^*) - \Pi(p)}{p^*\, y(p^*)} \approx \frac{\varepsilon - 1}{2}\, \xi^2.$$

(c) Now suppose all prices are fixed and the output gap is x_t. Show that the desired relative price moves by $\widehat{p_t^*} = \widehat{mc}_t = (\sigma + \varphi) x_t$, so that a firm that does not adjust loses

$$\frac{\varepsilon - 1}{2}\, (\sigma + \varphi)^2\, x_t^2$$

of its revenue. Compute this for $\varepsilon = 6$, $\sigma = \varphi = 1$ and $x_t = 5\%$, and compare with the welfare gain of Question 3.

Answer. (a) $\Pi'(p) = p^{-\varepsilon} Y - \varepsilon(p - mc) p^{-\varepsilon-1} Y$, which vanishes iff $p = \varepsilon(p - mc)$, i.e. $p^* = \mu\, mc$. Then $p^* - mc = p^*(1 - 1/\mu) = p^*/\varepsilon$, so $\Pi(p^*) = (p^*/\varepsilon)(p^*)^{-\varepsilon} Y = (p^*)^{1-\varepsilon} Y/\varepsilon$. Differentiating again, $\Pi''(p) = -2\varepsilon p^{-\varepsilon-1} Y + \varepsilon(\varepsilon+1)(p - mc) p^{-\varepsilon-2} Y$, and at p^*, using $(p^* - mc)/p^* = 1/\varepsilon$, $\Pi''(p^*) = p^{*\,-\varepsilon-1} Y[-2\varepsilon + \varepsilon + 1] = -(\varepsilon - 1) p^{*\,-\varepsilon-1} Y$. (b) By a second-order Taylor expansion around the optimum (the first-order term is zero – the envelope theorem), $\Pi(p^*) - \Pi(p) \approx -\frac{1}{2}\Pi''(p^*)(p - p^*)^2 = \frac{1}{2}(\varepsilon - 1)(p^*)^{1-\varepsilon} Y \xi^2$. Revenue at the optimum is $p^* y(p^*) = (p^*)^{1-\varepsilon} Y$, so the ratio is $\frac{1}{2}(\varepsilon - 1)\xi^2$. (c) The desired price is a constant markup over marginal cost, so $\widehat{p_t^*} = \widehat{mc}_t$, and from Question 1, $\widehat{mc}_t = \widehat{N^{\varphi} C^{\sigma}/Z} = (\sigma + \varphi)(y_t - y_t^n) = (\sigma + \varphi) x_t$ (Lecture 2). A firm that does not adjust has $\widehat{p}_t = 0$, so $\xi = -(\sigma + \varphi) x_t$ and the loss is $\frac{1}{2}(\varepsilon - 1)(\sigma + \varphi)^2 x_t^2$. Numbers: $\frac{1}{2} \times 5 \times 4 \times (0.05)^2 = 2.5\%$ of revenue. The social gain, from Question 3, is $x_t/\varepsilon = 5\%/6 = 0.83\%$ of consumption – and this is a gain to *every* household and firm, every period the gap persists, whereas the 2.5% is a one-off loss on one firm's revenue. The asymmetry becomes stark for smaller shocks: at $x_t = 1\%$ the private loss is 0.1% of revenue against a social gain of 0.17% of consumption. *Quadratic versus linear*: as $x_t \to 0$ the private incentive to adjust vanishes faster than the social gain, so arbitrarily small menu costs block adjustment while the output and welfare effects of money remain first order. *Remark on real rigidities*: both the private loss and the price gap are proportional to $(\sigma + \varphi)$, the elasticity of marginal cost to activity. If marginal cost is flat ($\sigma + \varphi$ small), firms have almost no reason to move their prices: price stickiness is then much easier to sustain and the Phillips curve is much flatter ($\kappa = \lambda(\sigma + \varphi)$). This is the NK counterpart of Blanchard–Kiyotaki's "$\alpha \to 1$" and of Ball & Romer's (1990) real rigidities.

Part C – Policy implications

6. Correcting the level distortion. Suppose the government pays firms an employment subsidy at rate τ, financed by lump-sum taxes, so that the effective marginal cost is $(1-\tau)W_t/(P_tZ_t)$. Show that

$$\tau = \frac{1}{\varepsilon} \qquad \Longleftrightarrow \qquad (1-\tau)\mu = 1 \qquad \Longrightarrow \qquad Y_t^n = Y_t^e, \quad \Phi = 0.$$

Answer. With the subsidy the firm's pricing rule becomes $P_t(i) = \mu\,(1-\tau)W_t/Z_t$, so in the symmetric equilibrium

$$mc_t \equiv \frac{W_t}{P_tZ_t} = \frac{1}{\mu(1-\tau)}.$$

Note that mc_t is measured *gross* of the subsidy, so it is still the social marginal cost – the resources the extra unit actually costs. Substituting the labor supply condition as in Question 1,

$$\frac{\left(Y_t^n\right)^{\sigma+\varphi}}{Z_t^{1+\varphi}} = \frac{N_t^{\varphi}C_t^{\sigma}}{Z_t} = mc_t = \frac{1}{\mu(1-\tau)},$$

so $Y_t^n = Y_t^e$ if and only if $\mu(1-\tau) = 1$, i.e. $\tau = 1 - 1/\mu = 1/\varepsilon$. At that point $mc_t = 1$: the bracket $1 - mc_t$ in Question 3 is zero and $\Phi = 0$. The subsidy exactly undoes the markup, so the price of a good equals its social marginal cost and there is no longer any first-order gain from producing more. The distortion is a *fiscal* one and is best cured with a fiscal instrument – which is why the NK literature studies monetary policy *conditional* on this subsidy being in place.

7. Welfare and the loss function. Lecture 2 states that, *with the subsidy in place*, a second-order approximation to welfare is

$$\mathbb{W} \approx -\frac{1}{2}\frac{\varepsilon}{\lambda}\,\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\Big[\pi_t^2 + \alpha_x\,x_t^2\Big], \qquad \alpha_x = \frac{\kappa}{\varepsilon}.$$

(a) Why does this expression contain no linear term? What changes without the subsidy?

(b) Why does inflation enter, given that price dispersion is of second order (Exercise 5)?

(c) Why is the relative weight on the output gap so small ($\alpha_x = \kappa/\varepsilon \approx 0.05$ for $\kappa = 0.3$, $\varepsilon = 6$)?

Answer. (a) With $\Phi = 0$ the linear term in x_t identified in Question 3 vanishes, and the leading term in the expansion is quadratic: zero inflation and zero gap is *the* optimum, and "strict inflation targeting" $\pi_t = 0$, $x_t = 0$ (implemented by $i_t = r_t^n$) is optimal. Without the subsidy, welfare contains a term $+\Phi\,\mathbb{E}_0\sum\beta^t x_t$ of *first* order: the policymaker's objective is then linear-quadratic with a positive linear coefficient, and the optimum involves a positive average output gap – which is exactly where the inflation bias of Exercise 7 comes from. (b) Inflation is costly *because* it implies price dispersion: when $\pi_t \neq 0$, firms that reset and firms that do not end up charging different prices, identical varieties are produced in different quantities, and the CES aggregator punishes that misallocation. Dispersion is second order in the state, which is why it can be ignored in a first-order approximation of the *equilibrium* (Exercise 5, Question 7), but welfare is itself a second-order object, so the same term is first order *in the loss*. The weight ε/λ is large: more substitutable varieties (high ε) make misallocation more costly, and stickier prices (low λ) make a given inflation rate correspond to more dispersion. (c) Multiplying through, the loss is proportional to $(\varepsilon/\lambda)\pi_t^2 + (\sigma+\varphi)x_t^2$: the utility cost of an output gap is governed by the curvature of the household's surplus, $(\sigma+\varphi)$, which is of order one, whereas the cost of inflation is scaled by ε/λ, which is one to two orders of magnitude larger ($\varepsilon = 6$, $\lambda = 0.17$ give $\varepsilon/\lambda \approx 35$, and stickier prices make it much larger). The ratio

is $\alpha_x = \lambda(\sigma + \varphi)/\varepsilon = \kappa/\varepsilon$. Many economists find this implausibly lopsided – it says a central bank should accept very large output gaps to avoid small amounts of inflation – which is one reason the literature has looked for additional costs of output gaps (unemployment, hysteresis, heterogeneous exposure to recessions) that this representative-agent model misses.

8. Discussion. What do the results of this exercise imply for

(a) the use of Calvo pricing as a modeling device?

(b) the output gap that a benevolent central bank would choose without the subsidy?

(c) the slope of the Phillips curve?

Answer. (a) The private cost of a wrong price is second order while the aggregate effect of money is first order, so *small* frictions – menu costs of a fraction of a percent of revenue – are enough to make nominal rigidity an equilibrium phenomenon. Calvo stickiness should be read as a tractable stand-in for this mechanism rather than as a literal technology, and the force of the argument is that one does not need large frictions for large real effects. (b) Positive. With $\Phi = 1/\varepsilon > 0$ the natural allocation is inefficiently depressed, so the central bank would like $x_t > 0$ on average. This is the microfoundation for the "$k < 1$" assumption in the Kydland–Prescott problem of Exercise 7 – and the reason the resulting inflation bias is best cured at the source, with the employment subsidy, rather than by monetary means. (c) The same elasticity $(\sigma + \varphi)$ that makes a non-optimal price costly also makes the Phillips curve steep, since $\kappa = \lambda(\sigma + \varphi)$. Real rigidities that flatten marginal cost therefore both sustain price stickiness and flatten the Phillips curve – which is the mechanism behind the empirical decomposition $\kappa = \lambda \times \Omega$ discussed in Lecture 2.

Exercise 7 – Time inconsistency and the inflation bias

Objective. By Exercise 6, without the employment subsidy welfare increases with the output gap at first order, so a benevolent policymaker wants a positive gap. Kydland & Prescott (1977) and Barro & Gordon (1983) show what follows. The optimal policy is not time consistent, and under discretion inflation is positive while output stays at its natural level.

Part A – The classic formulation

Consider an economy with an expectations-augmented Phillips curve

$$u = u_n - \alpha\,(\pi - \pi^e), \qquad \alpha > 0,$$

where u is unemployment, u_n its natural rate and π^e expected inflation. The government minimizes

$$L(u, \pi) = \frac{1}{2}\big(u - k\,u_n\big)^2 + \frac{1}{2}\gamma\,\pi^2, \qquad k < 1,\ \gamma > 0,$$

The government would like zero inflation and unemployment below the natural rate; γ is the weight on inflation relative to unemployment. Assume that the government controls π directly and that agents form expectations before the government moves.

1. Policy when agents expect zero inflation. Suppose agents expect $\pi^e = 0$. Show that the government chooses

$$\tilde{\pi} = \frac{\alpha}{\alpha^2 + \gamma}(1 - k)\,u_n > 0, \qquad \tilde{u} = \frac{\gamma + \alpha^2 k}{\gamma + \alpha^2}\,u_n < u_n.$$

Draw the problem in the (u, π) plane, with the indifference curves of L and the Phillips curve through $(u_n, 0)$. Is $(\tilde{\pi}, \tilde{u})$ a rational expectations equilibrium?

Answer. Minimizing L subject to $u = u_n - \alpha\pi$ (at $\pi^e = 0$), the first-order condition is

$$\frac{dL}{d\pi} = \gamma\pi + (u - ku_n)\frac{du}{d\pi} = \gamma\pi - \alpha(u - ku_n) = 0 \qquad \Longleftrightarrow \qquad \gamma\pi = \alpha\,(u - ku_n),$$

the familiar tangency between an indifference ellipse and the Phillips curve. Substituting the Phillips curve $u = u_n - \alpha\pi$: $\gamma\pi = \alpha[(1-k)u_n - \alpha\pi]$, which gives $\tilde{\pi}$, and then $\tilde{u} = u_n - \alpha\tilde{\pi} = \frac{\gamma+\alpha^2 k}{\gamma+\alpha^2}u_n$. Graphically: the government's bliss point is $(ku_n, 0)$, strictly to the left of $(u_n, 0)$; the indifference curves are ellipses centered there; the constraint is the downward-sloping Phillips curve through $(u_n, 0)$ with slope $-1/\alpha$ in the (u, π) plane. The optimum is the tangency point, which lies strictly *above* the horizontal axis. This is *not* a rational expectations equilibrium: agents who understand the government's incentives can predict $\tilde{\pi} > 0$, which contradicts $\pi^e = 0$. The announcement "we will keep inflation at zero" is not credible.

2. The rational expectations equilibrium. Solve the game by backward induction. Show that the government's best response to any π^e is

$$\pi = \frac{\alpha^2}{\gamma + \alpha^2}\,\pi^e + \frac{\alpha}{\gamma + \alpha^2}(1-k)u_n,$$

and deduce that the unique rational expectations equilibrium is

$$\pi^{RE} = \pi^e = \frac{\alpha}{\gamma}(1-k)\,u_n, \qquad u^{RE} = u_n.$$

Locate it on your graph.

Answer. The government's first-order condition $\gamma\pi = \alpha(u - ku_n)$ does not depend on π^e – the government takes expectations as given whatever they are. Substituting the Phillips curve $u = u_n - \alpha(\pi - \pi^e)$ gives $\gamma\pi = \alpha[(1-k)u_n - \alpha\pi + \alpha\pi^e]$, which rearranges to the stated best response: note the slope $\alpha^2/(\gamma + \alpha^2) \in (0, 1)$, so the government accommodates *part* of any rise in expected inflation. Rational expectations require $\pi^e = \mathbb{E}[\pi]$; with no uncertainty, $\pi^e = \pi$. Imposing this on the best response, $\pi^e\big[1 - \frac{\alpha^2}{\gamma+\alpha^2}\big] = \frac{\alpha}{\gamma+\alpha^2}(1-k)u_n$, and since the bracket equals $\gamma/(\gamma+\alpha^2)$, we get $\pi^e = \frac{\alpha}{\gamma}(1-k)u_n$. With $\pi = \pi^e$ the Phillips curve gives $u = u_n$. On the graph: the equilibrium is the point on the *vertical* line $u = u_n$ at which the government's best-response condition $\gamma\pi = \alpha(u - ku_n)$ holds, i.e. where the "$\gamma\pi = \alpha(1-k)u_n$" line crosses $u = u_n$. It lies strictly above $(\tilde{u}, \tilde{\pi})$: inflation is *higher* than in Question 1 and unemployment is *higher* too.

3. Feasible allocations and commitment.

(a) What is the set of allocations attainable under rational expectations? Which one does the government prefer?

(b) Why does it fail to implement it? Compute the welfare loss from discretion.

(c) Give two institutional arrangements that would remedy the problem. What does each one change in the model?

Answer. (a) With rational expectations and no uncertainty, $\pi = \pi^e$ always, so the Phillips curve collapses to $u = u_n$: the *long-run* aggregate supply curve is vertical and the feasible set is $\{(u_n, \pi) : \pi \in \mathbb{R}\}$. Since the government cannot affect unemployment, it prefers $\pi = 0$: the best feasible

allocation is $(u_n, 0)$. (b) The problem is *sequentiality*. To implement $(u_n, 0)$ agents must expect $\pi^e = 0$; but if they do, Question 1 shows the government strictly prefers to deviate to $\tilde{\pi} > 0$. So $(u_n, 0)$ is not sustainable as the outcome of the one-shot game: the optimal policy is not *time consistent*. The cost is pure: comparing the two feasible points,

$$L(u_n, \pi^{RE}) - L(u_n, 0) = \frac{\gamma}{2}\big(\pi^{RE}\big)^2 = \frac{\alpha^2(1-k)^2 u_n^2}{2\gamma} > 0,$$

inflation with no gain in unemployment whatsoever. Note it is *decreasing* in γ: a government that cares more about inflation suffers less from its own temptation – the key to the remedies. (c) *Rules rather than discretion*: commit to $\pi = 0$ by a mechanism that removes the government's period-by-period choice. In practice: (i) an *independent central bank* with a narrow mandate, i.e. a delegated loss function $L = \frac{1}{2}\pi^2$ (formally $\gamma \to \infty$, or $k \to 1$), under which $\pi^{RE} = 0$; (ii) a *conservative central banker* à la Rogoff (1985), $\gamma^{CB} > \gamma$, which reduces but need not eliminate the bias and – in a stochastic version – trades a lower bias against worse stabilization; (iii) *reputation* in a repeated game (Barro & Gordon, 1983), where the threat of losing credibility sustains low inflation; (iv) an explicit *inflation target* or contract for the central banker (Walsh, 1995); (v) removing the source of the temptation altogether – see Part B.

Part B – The same argument in the New Keynesian model

Restate the problem in terms of the output gap. The Phillips curve is $\pi_t = \pi_t^e + \kappa\, x_t$ with $\kappa = \lambda(\sigma+\varphi) > 0$, and the policymaker minimizes, period by period,

$$L = \frac{1}{2}\Big[\pi_t^2 + \alpha_x\big(x_t - x^*\big)^2\Big], \qquad \alpha_x = \frac{\kappa}{\varepsilon}, \qquad x^* > 0,$$

where x^* is the output gap the policymaker would like to run.

4. Discretionary equilibrium. Taking π_t^e as given, show that the policymaker's best response is $x_t = x^* - \kappa\pi_t/\alpha_x$, and that the rational expectations equilibrium is

$$x_t = 0, \qquad \pi_t = \frac{\alpha_x}{\kappa}\, x^* = \frac{x^*}{\varepsilon}.$$

Compare with Part A. How does the inflation bias depend on κ?

Answer. Substituting the Phillips curve into the loss and differentiating with respect to x_t, $\kappa\pi_t + \alpha_x(x_t - x^*) = 0$, i.e. $x_t = x^* - \kappa\pi_t/\alpha_x$. Under rational expectations $\pi_t^e = \pi_t$, so the Phillips curve forces $x_t = 0$, and the best response then gives $\pi_t = \alpha_x x^*/\kappa$; with $\alpha_x = \kappa/\varepsilon$ this is x^*/ε. This is Part A with $(u_n - u)/\alpha$ replaced by x_t and $(1-k)u_n$ by x^*: the structure is identical. Note the role of κ. In the *classic* formulation the bias $\frac{\alpha}{\gamma}(1-k)u_n$ is increasing in the slope α of the Phillips curve – a steeper trade-off makes the temptation stronger. Here, because α_x is itself proportional to κ, the two effects cancel and the bias x^*/ε is *independent* of κ: a flatter Phillips curve means the policymaker must create more inflation to buy a given gap, but it also means that any given inflation rate is less costly in terms of price dispersion. The bias depends only on x^* and on the elasticity of substitution.

5. The origin of x^*. Use Exercise 6 to argue that, without the employment subsidy, the welfare-based loss function contains a linear term in x_t whose coefficient is proportional to $\Phi = 1/\varepsilon$. Why does this imply $x^* > 0$? Which policy instrument removes the inflation bias?

Answer. Exercise 6, Question 3 showed that near the natural allocation a demand expansion raises welfare at first order, with weight $\Phi = 1 - 1/\mu = 1/\varepsilon$: the markup means price exceeds social marginal cost on every unit, so natural output is too low. Adding a linear term to the quadratic loss and completing the square turns $-\Phi x_t + \frac{\alpha_x}{2} x_t^2$ into $\frac{\alpha_x}{2}(x_t - x^*)^2$ plus a constant, with $x^* = \Phi/\alpha_x > 0$ (here Φ denotes the linear coefficient after the loss has been normalized as in Exercise 6, Question 7). So $k < 1$ in Part A is not an assumption about a politically motivated government: it is what a *fully benevolent* policymaker's preferences look like in a monopolistically competitive economy. The implication is important and is the main message of the policy section of Lecture 2: the inflation bias is a symptom of a *real* distortion, and the right instrument is the one that addresses it. An employment subsidy $\tau = 1/\varepsilon$ sets $\Phi = 0$, hence $x^* = 0$, hence $\pi^{RE} = 0$: the temptation disappears and strict inflation targeting becomes both optimal and time consistent. Using monetary policy to fix a fiscal distortion buys nothing in equilibrium – the only thing it achieves is inflation.

6. Discounting and the long-run trade-off. Questions 4 and 5 used a static Phillips curve. In the NK model the Phillips curve is $\pi_t = \beta\, \mathbb{E}_t \pi_{t+1} + \kappa x_t$.

(a) Show that in a stationary discretionary equilibrium (no shocks, $\mathbb{E}_t \pi_{t+1} = \pi_t$)

$$\pi_t = \frac{\kappa\, x^*}{1 - \beta + \kappa^2/\alpha_x} = \frac{\kappa\, x^*}{1 - \beta + \kappa\varepsilon}, \qquad x_t = \frac{(1-\beta)\, x^*}{1 - \beta + \kappa\varepsilon}.$$

(b) Check that $\beta \to 1$ gives the result of Question 4. Compute both expressions for $\kappa = 0.3$, $\varepsilon = 6$ and $\beta = 0.99$ (quarterly). Is the inflation bias increasing or decreasing in β? Why?

(c) For $\beta < 1$ the output gap is positive in equilibrium, because the NKPC implies the long-run relation $x = (1-\beta)\pi/\kappa$. How large is this trade-off for the parameters of (b)? Does it change the conclusion of Question 4?

Answer. (a) The policymaker takes $f_t \equiv \beta \mathbb{E}_t \pi_{t+1}$ as given – it cannot influence private expectations of *future* policy – and minimizes subject to $\pi_t = f_t + \kappa x_t$, so the first-order condition is unchanged: $x_t = x^* - \kappa \pi_t / \alpha_x$. Substituting into the Phillips curve and imposing the stationary rational expectation $f_t = \beta \pi_t$,

$$\pi_t = \beta \pi_t + \kappa x^* - \frac{\kappa^2}{\alpha_x} \pi_t \qquad \Longrightarrow \qquad \pi_t \Big[1 - \beta + \frac{\kappa^2}{\alpha_x} \Big] = \kappa x^*,$$

and $\kappa^2/\alpha_x = \kappa\varepsilon$ when $\alpha_x = \kappa/\varepsilon$. The gap follows from the Phillips curve itself: $\pi(1-\beta) = \kappa x$. (b) At $\beta = 1$: $\pi = \kappa x^*/(\kappa\varepsilon) = x^*/\varepsilon$ and $x = 0$, exactly Question 4 – the static formulation, in which rational expectations force $\pi^e = \pi$, *is* the $\beta = 1$ case. With $\kappa = 0.3$, $\varepsilon = 6$, $\beta = 0.99$: $\kappa\varepsilon = 1.8$ and $\pi = 0.3x^*/1.81 = 0.166\, x^*$ against $0.167\, x^*$ at $\beta = 1$, with $x = 0.0055\, x^*$. Quantitatively the two are indistinguishable. The bias is *increasing* in β: the more weight private agents put on expected future inflation, the more today's inflation is inherited from the anticipation of tomorrow's temptation, so the inability to commit compounds across periods. The static case is the upper bound. (c) Formally yes, arithmetically no. The implied long-run trade-off is $dx/d\pi = (1-\beta)/\kappa = 0.01/0.3 = 0.033$: a *permanent* increase in quarterly inflation of one percentage point (four points at an annual rate) buys a permanent output gap of three hundredths of a percentage point. Moreover this trade-off is an artefact of discounting the *utility* of future generations of price setters, not of any nominal rigidity, and it vanishes under indexation or at non-zero trend inflation. The honest reading is that the long-run Phillips curve is vertical for all practical purposes and the Kydland–Prescott conclusion stands: under discretion the economy ends up with inflation and no output. *Remark:* this is only the inflation-bias half of the commitment problem. The other half – the *stabilization bias* – survives even when $x^* = 0$: facing a cost-push shock u_t, a policymaker who can commit will promise *future*

disinflation in order to lower $\mathbb{E}_t \pi_{t+1}$ and thus moderate today's inflation at a smaller cost in output, a promise a discretionary policymaker cannot make. Both biases are reasons why "rules rather than discretion" remains the organizing principle of modern monetary frameworks.