The Master Equation, Projection, and Aggregation for Heterogeneous Agents models with Aggregate Risk

WORK IN PROGRESS

Thomas Bourany
COLUMBIA / EIEF

April 2025

Solving Heterogeneous Agents models with aggregate risk is hard

- Main difficulty with introducing risk in models with inequality/heterogeneity
  - With rational expectations (FIRE), agents need to forecast price/aggregate dynamics
  - ⇒ The distribution of agents g enters the household/firm decision problem
  - It gives rise to a "Master equation", the value depends on an infinite-dimensional object g

Solving Heterogeneous Agents models with aggregate risk is hard

- Main difficulty with introducing risk in models with inequality/heterogeneity
  - With rational expectations (FIRE), agents need to forecast price/aggregate dynamics
  - ⇒ The distribution of agents g enters the household/firm decision problem
  - It gives rise to a "Master equation", the value depends on an infinite-dimensional object g

- I provide a method to tackle and simplify this Master Equation
  - Merge Krusell-Smith's original idea of projection on moments with modern math treatments

Solving Heterogeneous Agents models with aggregate risk is hard

- Main difficulty with introducing risk in models with inequality/heterogeneity
  - With rational expectations (FIRE), agents need to forecast price/aggregate dynamics
  - ⇒ The distribution of agents g enters the household/firm decision problem
  - It gives rise to a "Master equation", the value depends on an infinite-dimensional object g

- I provide a method to tackle and simplify this Master Equation
  - Merge Krusell-Smith's original idea of projection on moments with modern math treatments
  - ▷ Provide applications with meaningful risk and non-linearity:
    - (i) Benchmark "non-rational expections" methods
    - (ii) Macro-Finance, asset pricing, and portfolio choice
    - (iii) Dynamic Oligopoly models

This project

- I suggest a new method (another one?) to solve HA models with aggregate risk

This project

- I suggest a new method (another one?) to solve HA models with aggregate risk
  - Huge literature since Krusell-Smith (KS) and Den Haan
  - State of the art – "MIT shocks", Perturbation methods, *Reiter (2009)*, *Ahn, Kaplan, Moll, Wolf (2018)*, *Bilal (2024)'s FAME*, Sequence Space methods *Auclert, Bardóczy, Rognlie, Straub (2021)*, Truncation methods, *Le Grand Ragot (2024)*, and others – rely mostly on certainty equivalence

This project

- I suggest a new method (another one?) to solve HA models with aggregate risk
  - Huge literature since Krusell-Smith (KS) and Den Haan
  - State of the art – “MIT shocks”, Perturbation methods, *Reiter (2009)*, *Ahn, Kaplan, Moll, Wolf (2018)*, *Bilal (2024)’s FAME*, Sequence Space methods *Auclert, Bardóczy, Rognlie, Straub (2021)*, Truncation methods, *Le Grand Ragot (2024)*, and others – rely mostly on certainty equivalence
  - More recent methods bypassing this limitation – Second-order methods, *Bhandari, Bourany, Evans, Golosov (2023), Bayer, Luetticke, Weiss, Winkelmann (2025), Bilal, Franco, Rossi-Hansberg (2026?)’s SAME*, Machine-Learning-based methods, *Fernandez-Villaverde, Hurtado, Nuno (2023), Huang (2023), Gu, Laurière, Merkel, Payne (2024), Yang, Wang, Schaab, Moll (2026)*, or others methods *Proehl (2019), Schaab (2021), Lee (2026)* – may seem case-specific, algebra heavy, or opaque

This project

- I suggest a new method to solve HA models with aggregate risk
  - State of the art methods rely mostly on certainty equivalence
  - More recent methods may seem a little opaque or case-specific

This project

- I suggest a new method to solve HA models with aggregate risk
  - State of the art methods rely mostly on certainty equivalence
  - More recent methods may seem a little opaque or case-specific
- I go back to the original idea of Krusell-Smith (1998)
  - I (still) summarize the distribution of agents with a finite set of moments: "projection"

This project

- I suggest a new method to solve HA models with aggregate risk
  - State of the art methods rely mostly on certainty equivalence
  - More recent methods may seem a little opaque or case-specific
- I go back to the original idea of Krusell-Smith (1998)
  - I (still) summarize the distribution of agents with a finite set of moments: "projection"
- But I borrow from the mathematics literature: Master equation
  - Mean Field Games and Master equation: *Cardaliaguet, Delarue, Lions, Lasry (2019)*
  - Also introduced in economics: *Schaab (2021), Bilal (2023), Gu, Laurière, Merkel, Payne (2024)*

This project

- I suggest a new method to solve HA models with aggregate risk
  - State of the art methods rely mostly on certainty equivalence
  - More recent methods may seem a little opaque or case-specific
- I go back to the original idea of Krusell-Smith (1998)
  - I (still) summarize the distribution of agents with a finite set of moments: "projection"
- But I borrow from the mathematics literature: Master equation
  - Mean Field Games and Master equation: *Cardaliaguet, Delarue, Lions, Lasry (2019)*
  - Also introduced in economics: *Schaab (2021), Bilal (2023), Gu, Laurière, Merkel, Payne (2024)*
- Can get additional insights on the nature of the equilibrium:
  - What are the drivers and amplification channels of aggregate fluctuations?
  - Are those restricted beliefs a good approximation for aggregation?
  - Applications with non-linearity and risk

Outline

Outline

1. **Krusell Smith Model**
2. Primer on the Master equation
3. Projection in Heterogeneous Agents Models
4. Numerical results for KS98
5. Testing bounded-rationality assumption in KS98.
6. “Macrofinance”: portfolio choice and Second Order Master Equation

Krusell-Smith (KS98) recap

- Consumption-saving model, c, a, with (i) idiosyncratic income risk z, (ii) incomplete market (iii) credit constraints $a \geq \underline{a}$, (iv) aggregate shock on aggregate TFP Z
- HH's distribution $g(a,z)$ over wealth a and income z

Krusell-Smith (KS98) recap

- Consumption-saving model, c, a, with (i) idiosyncratic income risk z, (ii) incomplete market (iii) credit constraints $a \geq \underline{a}$, (iv) aggregate shock on aggregate TFP Z
- HH's distribution $g(a,z)$ over wealth a and income z
- Firm side:

$$Y = ZK^{\alpha} \qquad \Rightarrow \qquad r = \alpha K^{\alpha-1} - \delta \qquad w = (1-\alpha)K^{\alpha}$$

- Household problem (KS98)

$$V(a,z,g,Z) = \max_{c,a'} \; u(c) + \beta \mathbb{E}^{z',Z'}\left[V(a',z',g',Z') \,\middle|\, z, Z\right]$$

$$s.t. \qquad c + a' = zw + (1+r)a$$

$$g' = H(g,Z,Z')$$

- Equilibrium

$$K = \int_{a,z} a \, g(da,z) \qquad \forall \, Z$$

General idea and KS98 global solution

- Difficulty:
  - Value $V(a, z, g, Z)$ depends on the whole distribution g
  - Need to forecast the evolution of g, possibly on *every path* of $\{Z_t\}_t$, *Achdou, Bourany (2018)*

General idea and KS98 global solution

- Difficulty:
  - Value $V(a, z, g, Z)$ depends on the whole distribution g
  - Need to forecast the evolution of g, possibly on *every path* of $\{Z_t\}_t$, *Achdou, Bourany (2018)*
- Krusell-Smith solution
  1. Assume Households only care about agg. capital / First-moment $K = \int a \, g(da, z)$
  2. Assume *Linear* forecasting rule:

$$\log K' = a_1^Z \log K + a_2^Z$$

  - Choose parameters (a_1^Z, a_2^Z) to match the *realized / simulated* path (Monte Carlo) of $\{K_t\}_t$

General idea and KS98 global solution

- Difficulty:
  - Value $V(a, z, g, Z)$ depends on the whole distribution g
  - Need to forecast the evolution of g, possibly on *every path* of $\{Z_t\}_t$, *Achdou, Bourany (2018)*
- Krusell-Smith solution
  1. Assume Households only care about agg. capital / First-moment $K = \int a \, g(da, z)$
  2. Assume *Linear* forecasting rule:

$$\log K' = a_1^Z \log K + a_2^Z$$

  - Choose parameters (a_1^Z, a_2^Z) to match the *realized / simulated* path (Monte Carlo) of $\{K_t\}_t$
- Proposal today: using the Master equation to:
  - remove assumption 2 $\Rightarrow$ bypass the linearity assumption $K' = b(K, Z)$
  - build macro dynamics $b(K, Z)$ from the *aggregation* of *micro* decisions
  - test robustness to 1 and 2, by adding more moments

Outline

Primer on Mean Field Games (MFG): the Aiyagari model

- States dynamics: saving and income shocks

$$da_t = [z_t w_t + r_t\, a_t - c_t]dt \qquad\qquad z_j \sim \text{Markov jump process } \lambda_j$$

Primer on Mean Field Games (MFG): the Aiyagari model

- States dynamics: saving and income shocks

$$da_t = [z_t w_t + r_t a_t - c_t]dt \qquad z_j \sim \text{Markov jump process } \lambda_j$$

1. Hamilton Jacobi Bellman Equation:

$$-\partial_t v(t,a,z) + \rho v(t,a,z) = \max_c u(c) + \mathcal{L}[\, v \mid c \,](t,a,z)$$

- Transport/Jump-Operator $\mathcal{L}$: *from agents' decision and shocks*

$$\mathcal{L}[\, v \,|c^\star](t,a,z_j) = \partial_a v(t,a,z_j) \underbrace{[z_j w + r\, a - c^\star]}_{\text{change in saving}} + \underbrace{\lambda_j \big(v(t,a,z_{-j}) - v(t,a,z_j)\big)}_{\text{change in labor income}}$$

Primer on Mean Field Games (MFG): the Aiyagari model

- States dynamics: saving and income shocks

$$da_t = [z_t w_t + r_t\, a_t - c_t]dt \qquad\qquad z_j \sim \text{Markov jump process } \lambda_j$$

1. Hamilton Jacobi Bellman Equation:

$$-\partial_t v(t,a,z) + \rho v(t,a,z) = \max_c\; u(c) + \mathcal{L}[\, v \mid c\,](t,a,z)$$

- Transport/Jump-Operator $\mathcal{L}$: *from agents' decision and shocks*

$$\mathcal{L}[\, v \mid c^\star](t,a,z_j) = \partial_a v(t,a,z_j) \underbrace{[z_j w + r\, a - c^\star]}_{\text{change in saving}} + \underbrace{\lambda_j\big(v(t,a,z_{-j}) - v(t,a,z_j)\big)}_{\text{change in labor income}}$$

2. Kolmogorov Forward Equation:

$$\partial_t g(t,a,z) = \mathcal{L}^*[\, g \mid c^\star](t,a,z)$$

  - $\mathcal{L}^*$: *distribution dynamics comes from agents' decisions*

- Equilibrium:

$$\iint_{z,a\geq \underline{a}} a\; g(t,da,z_j) = K \qquad\qquad r = \alpha K^{\alpha-1} - \delta$$

Primer on the Master Equation

- Master equation combines both HJB and KFE in *one equation*
  - No aggregate risk

$$-\partial_t v(t,a,z,g) + \rho v(t,a,z,g) = \overbrace{\max_c u(c) + \mathcal{L}[v\,|c^\star](t,a,z)}^{\text{standard HJB continuation value}} +$$

$$\underbrace{\iint_{z,a} \frac{\delta v(t,a,z,g)}{\delta g}[\tilde{a},\tilde{z}]\;\mathcal{L}^*[g\,|c^\star](t,\tilde{a},\tilde{z})}_{\text{change in } v \text{ due to the distribution dynamics}}$$

  - First part: HJB, how states (a,z) change agents' value v
  - Novelty: how the distribution g changes the value v
    - Forecast of agents (a,z) about *all* the other agents $(\tilde{a},\tilde{z})$

Primer on the Master Equation

- Master equation combines both HJB and KFE in *one equation*
  - No aggregate risk

$$-\partial_t v(t,a,z,g) + \rho v(t,a,z,g) = \overbrace{\max_c \, u(c) + \mathcal{L}[\, v \,|c^\star](t,a,z)}^{\text{standard HJB continuation value}} + \underbrace{\iint_{z,a} \frac{\delta v(t,a,z,g)}{\delta g}[\tilde{a},\tilde{z}] \;\, \mathcal{L}^*[\, g \,|c^\star](t,\tilde{a},\tilde{z})}_{\text{change in } v \text{ due to the distribution dynamics}}$$

  - First part: HJB, how states (a,z) change agents' value v
  - Novelty: how the distribution g changes the value v
    - Forecast of agents (a,z) about *all* the other agents $(\tilde{a},\tilde{z})$

    1. $\mathcal{L}^*[\, g \,|c^\star]$: Agents' decisions change the distribution g
    2. $\frac{\delta v}{\delta g}[\tilde{x}]$: How the distribution changes the value, when adding mass at $\tilde{x}$

Lions vs. Fréchet

Adding Aggregate Risk to the Master Equation

- Consider aggregate risk
  - Agg. TFP follows a AR(1) – Ornstein-Uhlenbeck process

$$dZ_t = -\theta(Z_t - \bar{Z})dt + \widehat{\sigma} dB_t^0$$

Adding Aggregate Risk to the Master Equation

- Consider aggregate risk
  - Agg. TFP follows a AR(1) – Ornstein-Uhlenbeck process

$$dZ_t = -\theta(Z_t - \bar{Z})dt + \widehat{\sigma} dB_t^0$$

  - Value $v = v(t,a,z,\boldsymbol{g},Z)$

$$-\partial_t v + \rho v = \overbrace{\max_c \; u(c) + \mathcal{L}[\,v\,|c](t,a,z)}^{\text{standard HJB continuation value}} \quad \overbrace{-\theta(Z-\bar{Z})v_Z + \frac{\widehat{\sigma}^2}{2} v_{ZZ}}^{\text{direct effect of risk of } Z \text{ on } v}$$

$$+ \underbrace{\iint_{z,a} \frac{\delta v(t,a,z,\boldsymbol{g},Z)}{\delta g}[\tilde{a},\tilde{z}] \;\; \mathcal{L}^*[\,g\,|c^\star](t,\tilde{a},\tilde{z})}_{\text{change due to distribution dynamics}}$$

Adding Aggregate Risk to the Master Equation

- Consider aggregate risk
  - Agg. TFP follows a AR(1) – Ornstein-Uhlenbeck process

$$dZ_t = -\theta(Z_t - \bar{Z})dt + \widehat{\sigma} dB_t^0$$

  - Value $v = v(t,a,z,g,Z)$

$$-\partial_t v + \rho v = \overbrace{\max_c \; u(c) + \mathcal{L}[\, v \,|c](t,a,z)}^{\text{standard HJB continuation value}} \quad \overbrace{-\theta(Z-\bar{Z})v_Z + \frac{\widehat{\sigma}^2}{2} v_{ZZ}}^{\text{direct effect of risk of } Z \text{ on } v}$$

$$+ \underbrace{\iint_{z,a} \frac{\delta v(t,a,z,g,Z)}{\delta g}[\tilde{a},\tilde{z}] \;\; \mathcal{L}^*[\, g \,|c^\star](t,\tilde{a},\tilde{z})}_{\text{change due to distribution dynamics}}$$

  - Master equation doesn't change much: Why?
    - No *direct effects* on individual states from agg. shocks: distribution g is not affected by dB_t^0
    - If it were (portfolio problems, see later): becomes *second order* $\Rightarrow$ Monster equation!

Outline

1. Krusell Smith Model
2. Primer on the Master equation
3. **Projection in Heterogeneous Agents Models**
4. Numerical results for KS98
5. Testing bounded-rationality assumption in KS98.
6. “Macrofinance”: portfolio choice and Second Order Master Equation
7. Dynamic Oligopoly: industry dynamics and Discrete Master Equation

Projection assumption in Krusell-Smith (KS98)

- What do households need for decisions? *only* changes in prices (r, w)

  $\Rightarrow$ Do not care of distribution g *per se*, Only one moment, K, ***the mean*** of g to forecast prices

$$K = \iint_{a,z} a\, g(da, z) \qquad\qquad r = \alpha K^{\alpha-1} - \delta$$

- Assumption: Restricted belief (here as in KS98)

$$v(a, z, g, Z) = \bar{v}(a, z, K^h, Z) + \varepsilon$$

  - Dimensionality reduction: $\mathbb{E}\left[\varepsilon \,\middle|\, g \ s.t.\ K = \iint a g(da, z)\right] = 0 \quad + \quad$ information loss

- Novelty: Nice property in Lions-derivative for the Master equation

$$\text{with } K^h = \int_x h(x)\, g(dx) \qquad\qquad \frac{\delta v(x, g)}{\delta g}[\tilde{x}] \equiv \frac{d\bar{v}(x, K^h)}{dK^h}\, h(\tilde{x})$$

Projection in the Master equation

- With first-moment: $v(a,z,g,Z) \approx \bar{v}(a,z,K,Z)$

$$\rho\bar{v} = \overbrace{\max_c u(c) + \mathcal{L}[\,\bar{v} \mid c\,]_{(a,z,K,Z)}}^{\text{standard HJB continuation value}} \overbrace{-\theta(Z-\bar{Z})\bar{v}_Z + \frac{\hat{\sigma}^2}{2}\bar{v}_{ZZ}}^{\text{direct effect of risk of } Z \text{ on } \bar{v}}$$

$$+\, \bar{v}_K \iint_{z,a} \underbrace{\left[r\tilde{a} + w\tilde{z} - c^\star(\tilde{a},\tilde{z},K,Z)\right]}_{\text{change in agents } (\tilde{a},\tilde{z}) \text{ decisions}} g(d\tilde{a},\tilde{z})$$

- Aggregation:

$$dK = \iint_{z,a} \left[r\tilde{a} + w\tilde{z} - c^\star(\tilde{a},\tilde{z},K,Z)\right] g(d\tilde{a},\tilde{z})$$

$$dK = rK + w\bar{L} - \mathcal{C}(K,Z|g) \qquad \text{w/ agg conso:} \quad \mathcal{C}(K,Z|g) = \textstyle\iint_{z,a} c^\star(\tilde{a},\tilde{z},K,Z)\; g(d\tilde{a},\tilde{z})$$

  - Micro-foundation of the law of motion dK through aggregation
  - But still dependence on g! How to “get rid of it”?

Master Equation becomes a fusion of two familiar HJB equations

- With $v = v(a, z, \boldsymbol{g}, Z) \approx \bar{v}(a, z, K, Z)$

$$\rho\,\bar{v} = \max_c\, u(c) + [wz + ra - c]\,\bar{v}_a + \lambda\big(\bar{v}_{(a,z',\cdot)} - \bar{v}_{(a,z,\cdot)}\big)$$

$$- \theta(Z - \bar{Z})\bar{v}_Z + \frac{\widehat{\sigma}^2}{2}\bar{v}_{ZZ} + \underbrace{\left[ZK^\alpha - \delta K - \mathcal{C}(K, Z|g)\right]}_{=dK}\,\bar{v}_K$$

Master Equation becomes a fusion of two familiar HJB equations

- With $v = v(a, z, g, Z) \approx \bar{v}(a, z, K, Z)$

$$\rho\,\bar{v} = \max_c\ u(c) + [wz + ra - c]\,\bar{v}_a + \lambda\big(\bar{v}(a,z',\cdot) - \bar{v}(a,z,\cdot)\big)$$

$$- \theta(Z - \bar{Z})\bar{v}_Z + \frac{\hat{\sigma}^2}{2}\bar{v}_{ZZ} + \underbrace{[ZK^\alpha - \delta K - \mathcal{C}(K, Z|g)]}_{=dK}\ \bar{v}_K$$

- Looks like a fusion of two standard models
  - RBC: $v = v(K, Z)$

$$\rho\,v = \max_C\ u(C) + [ZK^\alpha - \delta K - C]v_K - \theta(Z - \bar{Z})v_Z + \frac{\hat{\sigma}^2}{2}v_{ZZ}$$

  - Aiyagari: $v = v(a, z)$

$$\rho\,v = \max_c\ u(c) + [wz + ra - c]v_a + \lambda\big(v(a,z',\cdot) - v(a,z,\cdot)\big)$$

Agents' decision and global dynamical system

- With $v = \bar{v}(a, z, K, Z)$, we get individual decisions:

$$c^{\star}(\tilde{a}, \tilde{z}, K, Z) = \begin{cases} u'^{-1}\left(\bar{v}_a(\tilde{a}, \tilde{z}, K, Z)\right) \\ w(K,Z)z + r(K,Z)a \end{cases}$$

Agents' decision and global dynamical system

- With $v = \bar{v}(a, z, K, Z)$, we get individual decisions:

$$c^\star(\tilde{a}, \tilde{z}, K, Z) = \begin{cases} u'^{-1}\left(\bar{v}_a(\tilde{a}, \tilde{z}, K, Z)\right) \\ w(K,Z)z + r(K,Z)a \end{cases}$$

  - Thus, get a dynamical system for $x = (a, z, K, Z)$ — *Dynamical system*
  - Kolmogorov Forward equation for $x \Rightarrow$ Gives a distribution $\widetilde{g}(x)$ — *"Master" KFE*

Agents' decision and global dynamical system

- With $v = \bar{v}(a, z, K, Z)$, we get individual decisions:

$$c^{\star}(\tilde{a}, \tilde{z}, K, Z) = \begin{cases} u'^{-1}\left(\bar{v}_a(\tilde{a}, \tilde{z}, K, Z)\right) \\ w(K,Z)z + r(K,Z)a \end{cases}$$

  - Thus, get a dynamical system for $x = (a, z, K, Z)$ *Dynamical system*
  - Kolmogorov Forward equation for $x \Rightarrow$ Gives a distribution $\widetilde{g}(x)$ *"Master" KFE*
  - Easy to get $\widetilde{g}$ from Master Eq. operator $\mathcal{A}[\bar{v}|c^{\star}] \Leftrightarrow \mathcal{A}^{*}[\widetilde{g}|c^{\star}]$

Agents' decision and global dynamical system

- With $v = \bar{v}(a, z, K, Z)$, we get individual decisions:

$$c^\star(\tilde{a}, \tilde{z}, K, Z) = \begin{cases} u'^{-1}\left(\bar{v}_a(\tilde{a}, \tilde{z}, K, Z)\right) \\ w(K,Z)z + r(K,Z)a \end{cases}$$

  - Thus, get a dynamical system for $x = (a, z, K, Z)$ *Dynamical system*
  - Kolmogorov Forward equation for $x \quad \Rightarrow \quad$ Gives a distribution $\widetilde{g}(x)$ *"Master" KFE*
  - Easy to get $\widetilde{g}$ from Master Eq. operator $\mathcal{A}[\bar{v}|c^\star] \Leftrightarrow \mathcal{A}^*[\widetilde{g}|c^\star]$

- Consistency condition for rational-expectation equilibrium: $\sim$ *Bayes rules / Radon-Nykodim*

$$g(da, z_j)\Big|_{K,Z} = g\left(da, z_j \middle| K, Z\right) = \frac{\widetilde{g}(da, z_j, dK, dZ)}{\int_{\mathbb{X}} \widetilde{g}(da, z_j, dK, dZ)}$$

  - Using this g get $\mathcal{C}(K, Z|g) \quad \Rightarrow$ all we needed!!

Summary and numerical methods

1. General Master equation for $v(a, z, g, Z)$

2. Master Equation with "projection": $v = \bar{v}(a, z, K, Z)$
   - Start from guess $g(a,z)$, $\mathcal{C}(K, Z|g)$, and $dK^{(n)}$
   - Solve Master Equation (4 dims): finite difference methods
   - Get ind. decisions $c^\star(a,z,K,Z)$ and operator $\mathcal{A}[\bar{v}]$

3. "Master"-Kolmogorov forward for $x = (a,z,K,Z)$
   - Solve for distribution $\tilde{g}$ "for free" with $\mathcal{A}^*[\tilde{g}]$
   - Update g thanks to $\tilde{g}$ $\rightarrow$ update $\mathcal{C}(K, Z|g)$
   - Update agg. dynamics until convergence: $dK^{(n+1)} = ZK^\alpha - \delta K - \mathcal{C}(K, Z|g)$

Summary and numerical methods

1. General Master equation for $v(a, z, g, Z)$
2. Master Equation with "projection": $v = \bar{v}(a, z, K, Z)$
   - Start from guess $g(a,z)$, $\mathcal{C}(K, Z|g)$, and $dK^{(n)}$
   - Solve Master Equation (4 dims): finite difference methods
   - Get ind. decisions $c^{\star}(a,z,K,Z)$ and operator $\mathcal{A}[\bar{v}]$
3. "Master"-Kolmogorov forward for $x = (a,z,K,Z)$
   - Solve for distribution $\tilde{g}$ "for free" with $\mathcal{A}^*[\tilde{g}]$
   - Update g thanks to $\tilde{g}$ $\rightarrow$ update $\mathcal{C}(K, Z|g)$
   - Update agg. dynamics until convergence: $dK^{(n+1)} = ZK^{\alpha} - \delta K - \mathcal{C}(K, Z|g)$

- General procedure
  - No need for deep-learning/splines/polynomials: use "standard" finite difference methods
  - Aggregation from micro-decisions $\rightarrow$ potentially very non-linear!
  - Rational Expectation: Does not rely on bounded-rationality assumption of KS98

Master-Equation with higher moments:

- Extension with second moments: $v(a, z, g, Z) \approx \bar{v}(a, z, K, K_2, KL, Z)$
  - $K_2 = \mathbb{V}\text{ar}(a)$, $KL = \mathbb{C}\text{ov}(a, z)$.

$$\rho\,\bar{v} = \max_c u(c) + \mathcal{L}[\,\bar{v}\,|\,c\,] \; - \theta(Z - \bar{Z})\bar{v}_Z + \frac{\widehat{\sigma}^2}{2}\bar{v}_{ZZ} + \underbrace{\left(ZK^\alpha - \delta K - \mathbb{E}^g[c^\star]\right)}_{=dK}\,\bar{v}_K$$

$$+ \underbrace{\mathbb{C}\text{ov}^g(a, s^\star)}_{dK_2}\,\bar{v}_{K_2} \; - (\lambda_1 + \lambda_2)\,\underbrace{\mathbb{C}\text{ov}^g(a, z)}_{dKL \propto -KL}\,\bar{v}_{KL}$$

  - System: $x = (a, z, K, K_2, KL, Z)$: $\bar{v}(x) \Rightarrow c^\star(x)$ and $dK = \mathcal{S}(K, K_2, KL, Z|g)dt$

Master-Equation with higher moments:

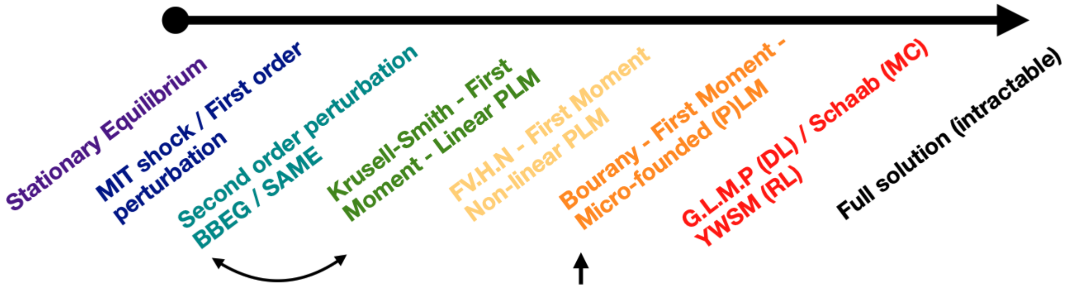
- Extension with second moments: $v(a,z,g,Z) \approx \bar{v}(a,z,K,K_2,KL,Z)$
  - $K_2 = \mathbb{V}\text{ar}(a)$, $KL = \mathbb{C}\text{ov}(a,z)$.

$$\rho\,\bar{v} = \max_c u(c) + \mathcal{L}[\,\bar{v}\,|\,c\,] - \theta(Z-\bar{Z})\bar{v}_Z + \frac{\widehat{\sigma}^2}{2}\bar{v}_{ZZ} + \underbrace{\left(ZK^\alpha - \delta K - \mathbb{E}^g[c^\star]\right)}_{=dK}\bar{v}_K$$

$$+ \underbrace{\mathbb{C}\text{ov}^g(a,s^\star)}_{dK_2}\bar{v}_{K_2} - (\lambda_1+\lambda_2)\underbrace{\mathbb{C}\text{ov}^g(a,z)}_{dKL\propto -KL}\bar{v}_{KL}$$

  - System: $x=(a,z,K,K_2,KL,Z)$: $\bar{v}(x) \Rightarrow c^\star(x)$ and $dK = \mathcal{S}(K,K_2,KL,Z|g)dt$
  - Theoretical insights: (numerics WIP)
  - With more moments (= information), one predict more accurately price dynamics (r,w)
  - Distribution over moments $\widetilde{g}(x)$: self-regulating forces (e.g. KL)
  - If $\mathbb{C}\text{ov}^g(a,s^\star)>0 \;\Rightarrow\; K_2\uparrow$, and if $\bar{v}_{K_2}<0$, $\rightarrow$ reinforces precautionary saving

Summary and comparison



Outline

1. Krusell Smith Model
2. Primer on the Master equation
3. Projection in Heterogeneous Agents Models
4. **Numerical results for KS98**
5. Testing bounded-rationality assumption in KS98.
6. “Macrofinance”: portfolio choice and Second Order Master Equation
7. Dynamic Oligopoly: industry dynamics and Discrete Master Equation

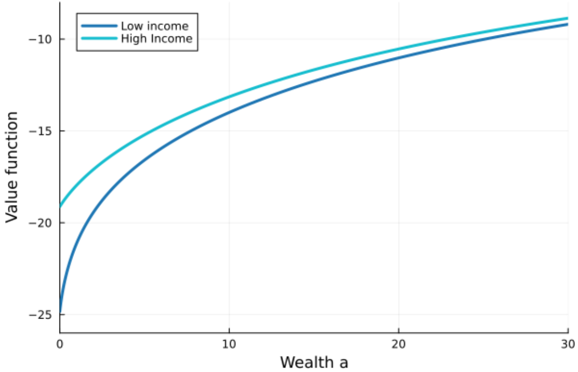
Numerical results – Summary

- Risk:
  - Idiosyncratic risk: Two-state Markov process for labor income shocks z
  - Aggregate risk: Three-state Markov process for TFP Z, with $\sigma(Z) = 12\%$
- I compare three economies:

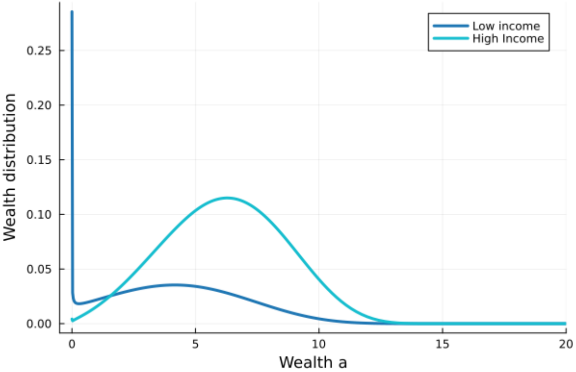
1. RBC (Brock Mirman) model
   $\Rightarrow$ (v, g) value and distribution over aggregate capital and TFP (K, Z).
2. Aiyagari model
   $\Rightarrow$ (v, g) individual heterogeneity on (a, z), for constant TFP $Z = \bar{Z}$ and capital $K = \bar{K}$.
3. Krusell-Smith model
   - Have both: (v, g) over (a, z, K, Z)
   - Iterate over $dK = ZK^{\alpha} - \delta K - \mathcal{C}(K, Z|g)$

Recap – Aiyagari model

Value function $v(a,z)$

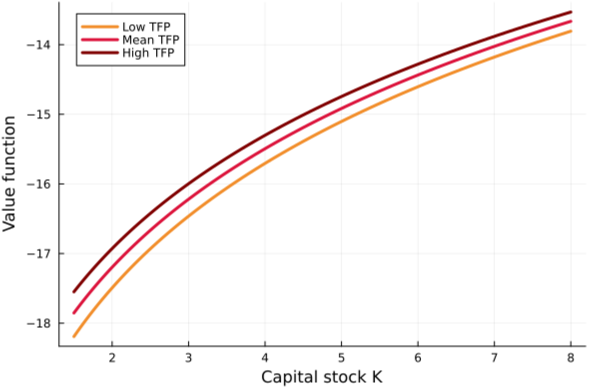


Distribution $g(a,z)$

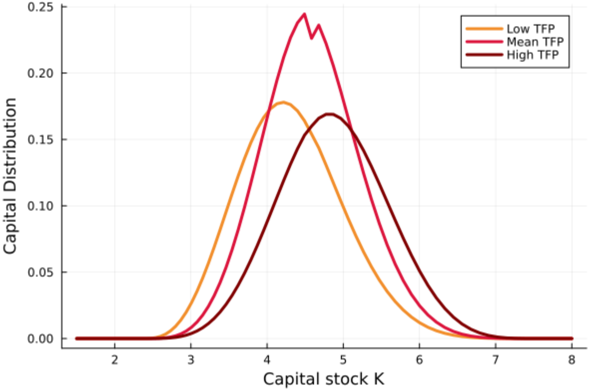


Recap – Brock-Mirman / RBC

Value function $v(K,Z)$

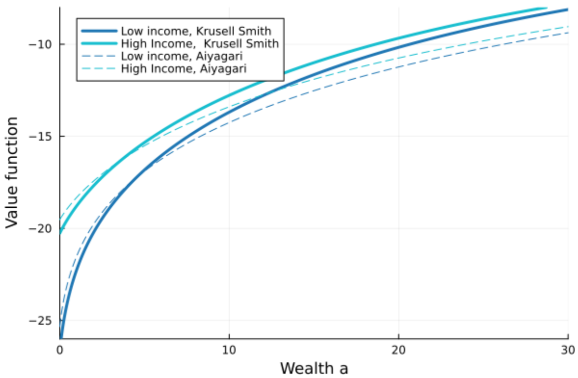


Distribution $g(K,Z)$

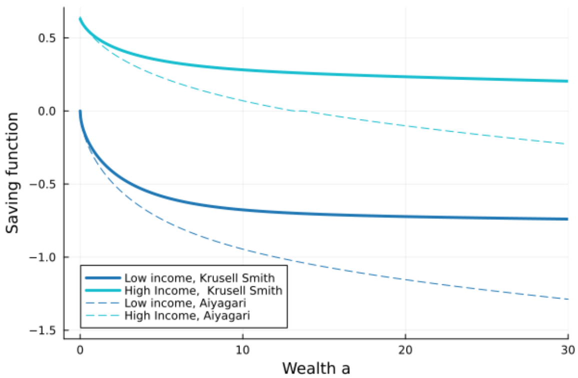


Krusell-Smith: individual decisions, vs. Aiyagari

Value function $v(a, z, \bar{K}, \bar{Z})$

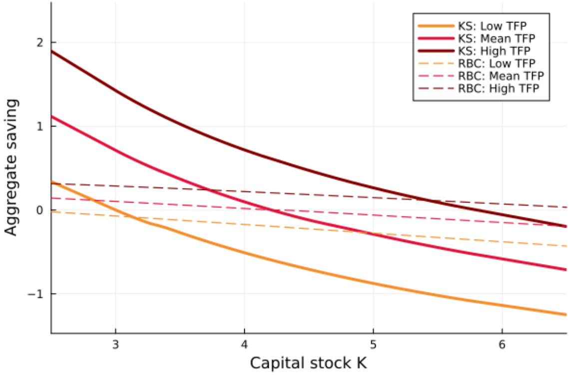


Saving $s(a, z, \bar{K}, \bar{Z}) = wz + ra - c^{\star}$



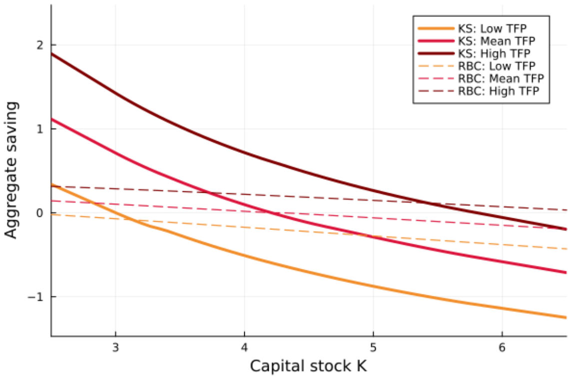
Krusell-Smith: aggregate dynamics decisions, vs. RBC

Aggregate dynamics $dK = ZK^{\alpha} - \delta K - \mathcal{C}(K, Z|g)$

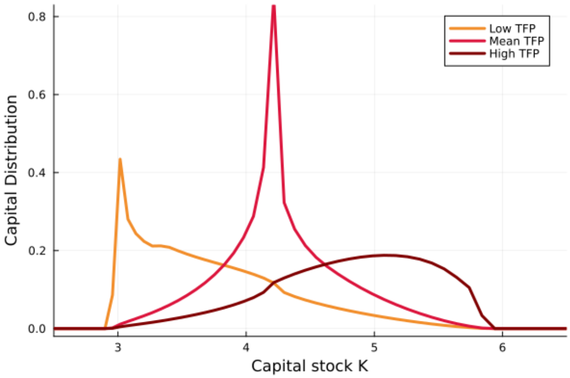


Krusell-Smith: aggregate dynamics decisions, vs. RBC

Aggregate dynamics $dK = ZK^{\alpha} - \delta K - \mathcal{C}(K, Z|g)$

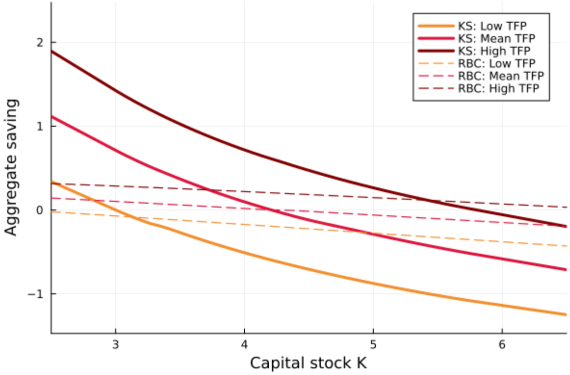


Distribution $\tilde{g}(K, Z)$

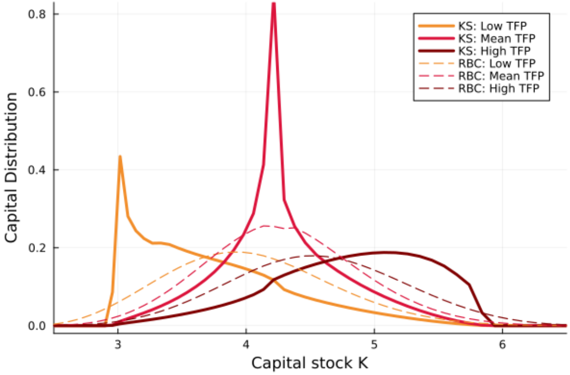


Krusell-Smith: aggregate dynamics decisions, vs. RBC

Aggregate dynamics $dK = ZK^{\alpha} - \delta K - \mathcal{C}(K, Z|g)$



Distribution $\tilde{g}(K, Z)$



Effect of aggregate uncertainty and heterogeneity

1. Higher individual precautionary saving motives:
   - Aggregate uncertainty affects the poor (reliant on labor income) more than the rich (hedging: $r = MPK \uparrow$ when $K \downarrow$)
   - Implies much more savings when rich: flatter savings function compared to Aiyagari

Effect of aggregate uncertainty and heterogeneity

1. Higher individual precautionary saving motives:
   - Aggregate uncertainty affects the poor (reliant on labor income) more than the rich (hedging: $r = MPK \uparrow$ when $K \downarrow$)
   - Implies much more savings when rich: flatter savings function compared to Aiyagari
2. Aggregate dynamics are more "reactive"
   - Individual heterogeneity implies a steeper aggregate saving function
   - Rich household save a lot more when K is lower, boosting investment

Effect of aggregate uncertainty and heterogeneity

1. Higher individual precautionary saving motives:
   - Aggregate uncertainty affects the poor (reliant on labor income) more than the rich (hedging: $r = MPK \uparrow$ when $K \downarrow$)
   - Implies much more savings when rich: flatter savings function compared to Aiyagari
2. Aggregate dynamics are more "reactive"
   - Individual heterogeneity implies a steeper aggregate saving function
   - Rich household save a lot more when K is lower, boosting investment
3. Change the distribution of economic activity
   - Output is fluctuating more in Krusell-Smith: 15.5% volatility compared to 14% for RBC
   - However, business cycles are less skewed – smaller right tail, more symmetric – and have less kurtosis – thinner tails of capital/output.

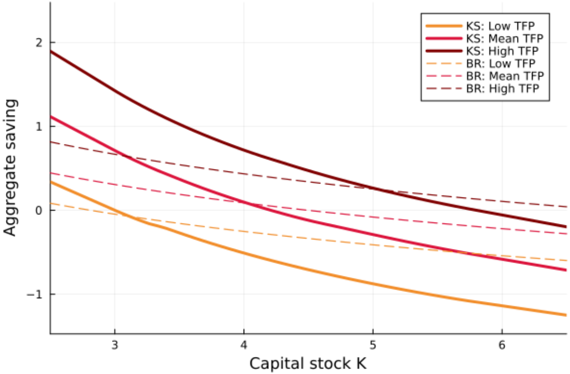
Outline

1. Krusell Smith Model
2. Primer on the Master equation
3. Projection in Heterogeneous Agents Models
4. Numerical results for KS98
5. **Testing bounded-rationality assumption in KS98**
6. “Macrofinance”: portfolio choice and Second Order Master Equation

Bounded-rationality in Krusell-Smith

- Agents do not forecast the "true" capital dynamics dK
  - Assume a log-linear rule: $dK = \beta^Z \log(K)dt$
  - What do we "lose" from this assumption?

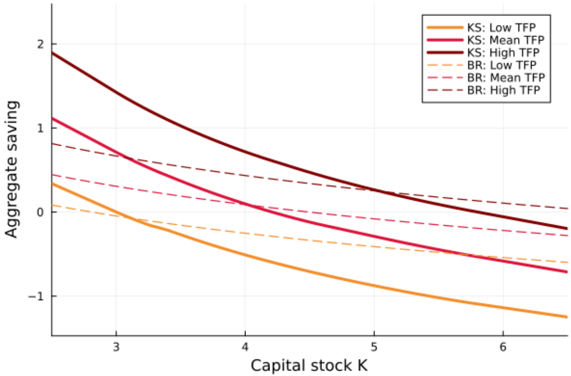
Aggregate dynamics $dK = \beta^Z \log(K)dt$



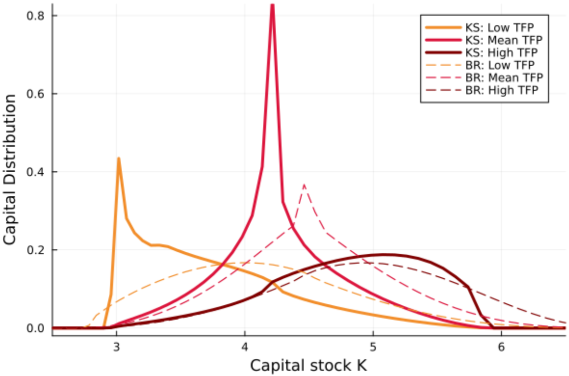
Bounded-rationality in Krusell-Smith

- Agents do not forecast the "true" capital dynamics dK
  - Assume a log-linear rule: $dK = \beta^Z \log(K) dt$
  - What do we "lose" from this assumption?

Aggregate dynamics $dK = \beta^Z \log(K) dt$



Distribution $\tilde{g}(K, Z)$



Outline

Master equation and Merton portfolio problem

- Agg. risk: both TFP (OU) $Z \sim \widehat{\sigma} dB_t^0$ and *direct effect* on portfolio return (share α)

$$da = (ra + zw - c)dt + \alpha\, a\, (r^k - r)dt + \alpha\, a\, \overline{\sigma} dB_t^0$$

  - The master equation now becomes ***second order***! value $v = v_{(a,z,g,Z)}$ changes a lot!

$$\rho v = \overbrace{\max_{c,\alpha}\; u(c) + \mathcal{L}[v|c,\alpha]_{(a,z,g,Z)}}^{\text{standard HJB continuation value}} \overbrace{-\theta(Z-\bar{Z})v_Z + \frac{\widehat{\sigma}^2}{2} v_{ZZ}}^{\text{direct effect of risk of } Z \text{ on } v} + \overbrace{\iint_{z,a} \frac{dv_{(a,z,g,Z)}}{dg}[\tilde{a},\tilde{z}]\; \mathcal{L}^*[\,g\,|c^\star]_{(\tilde{a},\tilde{z})}}^{\text{deterministic evolution of the distribution}}$$

Master equation and Merton portfolio problem

- Agg. risk: both TFP (OU) $Z \sim \widehat{\sigma} dB_t^0$ and *direct effect* on portfolio return (share α)

$$da = (ra + zw - c)dt + \alpha\, a\, (r^k - r)dt + \alpha\, a\, \overline{\sigma} dB_t^0$$

  - The master equation now becomes ***second order***! value $v = v_{(a,z,\boldsymbol{g},Z)}$ changes a lot!

$$\rho v = \overbrace{\max_{c,\alpha} u(c) + \mathcal{L}[v|c,\alpha]_{(a,z,\boldsymbol{g},Z)}}^{\text{standard HJB continuation value}} \overbrace{-\theta(Z - \bar{Z})v_Z + \frac{\widehat{\sigma}^2}{2} v_{ZZ}}^{\text{direct effect of risk of } Z \text{ on } v} + \overbrace{\iint_{z,a} \frac{dv_{(a,z,\boldsymbol{g},Z)}}{dg}[\tilde{a},\tilde{z}]\, \mathcal{L}^*[\, g\, |c^\star]_{(\tilde{a},\tilde{z})}}^{\text{deterministic evolution of the distribution}}$$

$$+ \underbrace{\frac{\overline{\sigma}^2}{2} \iint_{z,a} \alpha_{(\tilde{a},\tilde{z})}{}^2 \tilde{a}^2 \frac{d}{d\tilde{a}} \Big(\frac{dv}{dg}[\tilde{a},\tilde{z}] \Big) g_{(d\tilde{a},\tilde{z})}}_{\text{diffusion of the distribution due to risk}} + \underbrace{\overline{\sigma}^2 \alpha_{(a,z)} \iint_{z,a} \alpha_{(\tilde{a},\tilde{z})} \tilde{a} \frac{d}{da} \frac{dv}{dg}[\tilde{a},\tilde{z}]\, g_{(d\tilde{a},\tilde{z})}}_{\text{covariance of own state } a \text{ and distribution } \tilde{a}}$$

Master equation and Merton portfolio problem

- Agg. risk: both TFP (OU) $Z \sim \widehat{\sigma} dB_t^0$ and *direct effect* on portfolio return (share α)

$$da = (ra + zw - c)dt + \alpha\, a\, (r^k - r)dt + \alpha\, a\, \overline{\sigma} dB_t^0$$

  - The master equation now becomes ***second order***! value $v = v(a,z,g,Z)$ changes a lot!

$$\rho v = \overbrace{\max_{c,\alpha} u(c) + \mathcal{L}[v|c,\alpha](a,z,g,Z)}^{\text{standard HJB continuation value}} \overbrace{-\theta(Z-\bar{Z})v_Z + \frac{\widehat{\sigma}^2}{2} v_{ZZ}}^{\text{direct effect of risk of } Z \text{ on } v} + \overbrace{\iint_{z,a} \frac{dv(a,z,g,Z)}{dg}[\tilde{a},\tilde{z}]\, \mathcal{L}^*[g\,|c^\star](\tilde{a},\tilde{z})}^{\text{deterministic evolution of the distribution}}$$

$$+ \underbrace{\frac{\overline{\sigma}^2}{2} \iint_{z,a} \alpha(\tilde{a},\tilde{z})^2 \tilde{a}^2 \frac{d}{d\tilde{a}}\left(\frac{dv}{dg}[\tilde{a},\tilde{z}]\right) g(d\tilde{a},\tilde{z})}_{\text{diffusion of the distribution due to risk}} + \underbrace{\overline{\sigma}^2 \alpha(a,z) \iint_{z,a} \alpha(\tilde{a},\tilde{z})\tilde{a} \frac{d}{da}\frac{dv}{dg}[\tilde{a},\tilde{z}]\, g(d\tilde{a},\tilde{z})}_{\text{covariance of own state } a \text{ and distribution } \tilde{a}}$$

$$+ \underbrace{\overline{\sigma}\widehat{\sigma} \iint_{z,a} \alpha(\tilde{a},\tilde{z}) \frac{d}{dZ}\frac{dv}{dg}[\tilde{a},\tilde{z}]\, g(d\tilde{a},\tilde{z})}_{\text{covariance of agg. state } Z \text{ and distribution } \tilde{a}} + \underbrace{\frac{\overline{\sigma}^2}{2} \iint_{(z,a)^{\otimes 2}} \alpha(\tilde{a},\tilde{z})\alpha(\tilde{a}',\tilde{z}') \frac{d^2\overline{v}}{dg^2}[\tilde{a},\tilde{z},\tilde{a}',\tilde{z}']\, g(d\tilde{a},\tilde{z})\, g(d\tilde{a}',\tilde{z}')}_{\text{covariance of distribution } \tilde{a} \text{ and } \tilde{a}'}$$

Master equation with projection for portfolio problem

- Agg. risk: both TFP (OU) $Z \sim \widehat{\sigma} dB_t^0$ and *direct effect* on portfolio return (share α)
  - With projection $v(a,z,g,Z) = \bar{v}(a,z,K,Z)$, master equation simplifies:

$$\rho \bar{v} = \overbrace{\max_{c,\alpha} u(c) + \mathcal{L}[\bar{v}|c,\alpha](a,z,K,Z)}^{\text{standard HJB continuation value}} \overbrace{-\theta(Z-\bar{Z})\bar{v}_Z + \frac{\widehat{\sigma}^2}{2}\bar{v}_{ZZ}}^{\text{direct effect of risk of } Z \text{ on } v} + \overbrace{\left(ZK^\alpha - \delta K - \mathcal{C}(K,Z|g)\right)\bar{v}_K}^{\text{deterministic evolution of the distribution}}$$

$$+ \underbrace{\frac{\overline{\sigma}^2}{2} K \bar{v}_K \times 0}_{\text{diffusion of distribution due to risk } = 0} + \underbrace{\overline{\sigma}^2 \alpha(a,z,K,Z) a K \bar{v}_{Ka}}_{\text{covariance of own state } a \text{ and moment } K}$$

$$+ \underbrace{\overline{\sigma}\,\widehat{\sigma}\, K \bar{v}_{KZ}}_{\text{covar. of agg. state } Z \text{ and moment } K} + \underbrace{\frac{\overline{\sigma}^2}{2} K^2 \bar{v}_{KK}}_{\mathbb{V}\text{ar moment } K} \quad \text{with} \quad K = A(K,Z|g) = \iint_{z,a} \alpha(\tilde{a},\tilde{z},\cdot)\, \tilde{a}\, g(d\tilde{a},\tilde{z})$$

Optimal decisions and portfolio

- Agg. risk: both TFP (OU) $Z \sim \widehat{\sigma} dB_t^0$ and *direct effect* on portfolio return (share α)
  - With projection $v(a,z,g,Z) = \bar{v}(a,z,K,Z)$
  - Optimal decisions: consumption

$$c^\star(a, z_j, K, Z) = \begin{cases} u'^{-1}\left(\bar{v}_a(a,z_j,K,Z)\right) & a > \underline{a} \\ z_j\, w(K,Z) + r\, a & \text{if } a \leq \underline{a} \end{cases}$$

  - Optimal portfolio choice:

$$\alpha^\star(a, z_j, K, Z) = \frac{r^k - r}{\gamma \overline{\sigma}^2} + \frac{\widehat{\sigma}}{\overline{\sigma}\gamma} \frac{\bar{v}_{aZ}}{\bar{v}_a} + \frac{K\, \bar{v}_{aK}}{\gamma\, \bar{v}_a} \qquad \text{if} \quad a > \underline{a}$$

and $\alpha^\star(a, z_j, K, Z) = 0$ if $a \leq \underline{a}$ with $\frac{1}{\gamma(x)} = -\frac{\overline{V}_a(a,z_j,K,Z)}{\overline{V}_{aa}(a,z_j,K,Z)a}$

Outline

Dynamic Oligopoly – Ericson-Pakes

- Firms i with cost state $x \in \mathbb{X}$, investment a with inv. cost $c(x)a$
  - x drift $\mu(x, a)$, jump up w/ proba $\phi(x,a)$, fall w/ proba δ + entry / exit
- Market state $\boldsymbol{x} = \{x_i\}_i \Leftrightarrow$ emp. distribution of firms $g = \sum_i \delta_{x_i}$.

Dynamic Oligopoly – Ericson-Pakes

- Firms i with cost state $x \in \mathbb{X}$, investment a with inv. cost $c(x)a$
  - x drift $\mu(x, a)$,   jump up w/ proba $\phi(x,a)$,   fall w/ proba δ + entry / exit
- Market state $\boldsymbol{x} = \{x_i\}_i \Leftrightarrow$ emp. distribution of firms $g = \sum_i \delta_{x_i}$.
- Aggregate cost Z, following AR(1)-OU process $\widehat{\sigma}$
- Competition à la Bertrand, CES pref. σ
- Market share $s_i = \left(\frac{p_i}{P}\right)^{1-\sigma}$, and elasticity $\varepsilon_{i(s_i)} = \sigma - (\sigma - 1)s_i$

Dynamic Oligopoly – Ericson-Pakes

- Firms i with cost state $x \in \mathbb{X}$, investment a with inv. cost $c(x)a$
  - x drift $\mu(x, a)$, jump up w/ proba $\phi(x,a)$, fall w/ proba δ + entry / exit
- Market state $\boldsymbol{x} = \{x_i\}_i \Leftrightarrow$ emp. distribution of firms $g = \sum_i \delta_{x_i}$.
- Aggregate cost Z, following AR(1)-OU process $\widehat{\sigma}$
- Competition à la Bertrand, CES pref. σ
- Market share $s_i = \left(\frac{p_i}{P}\right)^{1-\sigma}$, and elasticity $\varepsilon_i(s_i) = \sigma - (\sigma - 1)s_i$

$\Rightarrow$ Firm profit $\pi(x_i, \boldsymbol{x}, Z) = \frac{E s_i}{\sigma - (\sigma - 1)s_i} - f$

- Static optimal pricing $p_i = \frac{\varepsilon_i}{\varepsilon_i - 1} x_i Z$
- Interaction in oligopolistic markets: price index $P^{1-\sigma} = \sum_i p_i^{1-\sigma}$

Dynamic Oligopoly – Master Equation

- Master equation: $v = v(x_i, g, Z)$, with a discrete distribution / finite states

$$\rho v = \overbrace{\max_a \pi(x_i, g, Z) - c(x_i) a + \mathcal{L}[v|a]}^{=\text{standard HJB cont. value}} \overbrace{-\theta(Z - \bar{Z})\bar{v}_Z + \frac{\widehat{\sigma}^2}{2} v_{ZZ}}^{=\text{direct effect of agg. risk } Z} + \overbrace{\int_x \frac{\delta v}{\delta g}[\tilde{x}]\ \mathcal{L}^*[g|a^\star](d\tilde{x})}^{=\text{evolution of the distribution}}$$

  - Operator $\mathcal{L}[v|a] = \mu_{(x,a)}\, v_x + \phi_{(x,a)}\big(v_{(x_{i+1},\cdot)} - v_{(x_i,\cdot)}\big) + \delta\big(v_{(x_{i-1},\cdot)} - v_{(x_i,\cdot)}\big)$

Dynamic Oligopoly – Master Equation

- Master equation: $v = v(x_i, g, Z)$, with a discrete distribution / finite states

$$\rho v = \overbrace{\max_a \pi(x_i, g, Z) - c(x_i)a + \mathcal{L}[v|a]}^{=\text{standard HJB cont. value}} \overbrace{-\theta(Z-\bar{Z})\bar{v}_Z + \frac{\widehat{\sigma}^2}{2} v_{ZZ}}^{=\text{direct effect of agg. risk } Z} + \overbrace{\int_x \frac{\delta v}{\delta g}[\tilde{x}] \; \mathcal{L}^*[g|a^\star](d\tilde{x})}^{=\text{evolution of the distribution}}$$

  - Operator $\mathcal{L}[v|a] = \mu(x,a)\, v_x + \phi(x,a)\Big(v(x_{i+1},\cdot) - v(x_i,\cdot)\Big) + \delta\Big(v(x_{i-1},\cdot) - v(x_i,\cdot)\Big)$

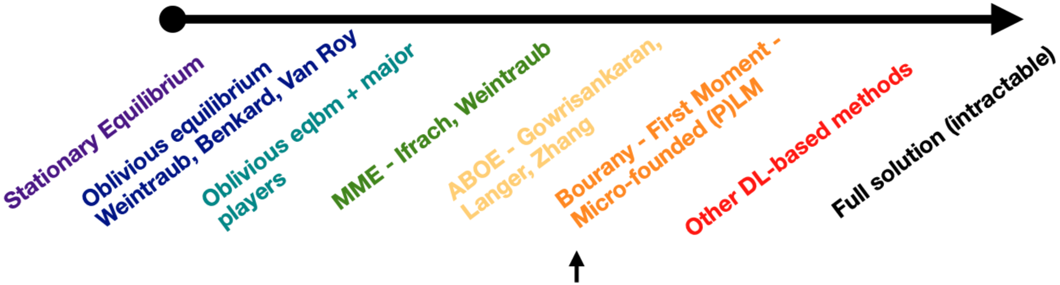
- Payoff-relevant agg. state: P. Projection $v(x_i, g, Z) \approx v(x_i, P, Z)$

$$\rho \bar{v} = \overbrace{\max_a \pi(x_i, g, Z) - c(x_i)a + \mathcal{L}[\bar{v}|a]}^{=\text{standard HJB cont. value}} \overbrace{-\theta(Z-\bar{Z})\bar{v}_Z + \frac{\widehat{\sigma}^2}{2} \bar{v}_{ZZ}}^{=\text{direct effect of agg. risk } Z} + \bar{v}_P \overbrace{\int_x \Big(\tfrac{p(x_j,P)}{P}\Big)^{-\sigma} \mu(x_i,a^\star) g(dx)}^{=\mathbb{C}\text{ov}^g\left(s_i^{\frac{\sigma}{\sigma-1}}, \mu(x,a)\right)}$$

$$+ \sum_j \hat{\Phi}_j^g \big(\bar{v}_P(\hat{P}^j) - \bar{v}_P\big) + \sum_j \check{\Phi}_j^g \big(\bar{v}_P(\check{P}^j) - \bar{v}_P\big)$$

with $\hat{P}^i = \Big(P^{1-\sigma} + \frac{1}{N}(p_{i+1}^{1-\sigma} - p_i^{1-\sigma})\Big)^{\frac{1}{1-\sigma}}$, firm i switch up, with proba $\hat{\Phi}_i^g = \phi(x_i, a_i^\star) g(x_i)$

Summary and comparison



Conclusion

- I propose a new method for solving Heterogeneous Agent Models with aggregate risk
  - Leverage the Master equation approach and generalize Krusell-Smith insights with projections on finite sets of moments
- Next steps:
  - Additional example of information frictions models
  - Macro-finance model to the test + Replicate *Fernandez-Villaverde, Hurtado, Nuno (2023 Ecma)*
  - Oligopoly model a la Ericson Pakes (1995) with a distribution of firms within markets

Primer on the Lions derivative

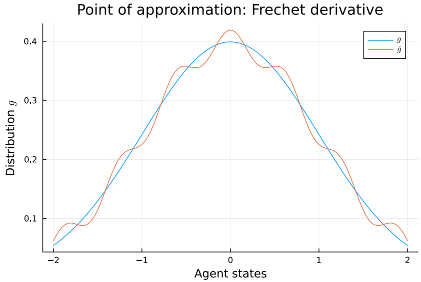
- Derivative in the space of distribution: how the value $v(a,z,\boldsymbol{g})$ changes when the distribution of agents $\boldsymbol{g}$ moves?

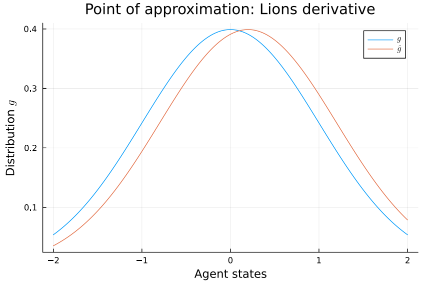
$$dv(a,z,\boldsymbol{g}) \approx v(a,z,\tilde{\boldsymbol{g}}) - v(a,z,\boldsymbol{g})$$

$$\approx \iint_{\tilde{a},\tilde{z}} \underbrace{\frac{\partial v(a,z,\boldsymbol{g})}{\partial g}[\tilde{a},\tilde{z}]}_{=\text{Fréchet}} \left(\tilde{g}(\tilde{a},\tilde{z}) - g(\tilde{a},\tilde{z})\right)$$

$$\approx \iint_{\tilde{a},\tilde{z}} \underbrace{\frac{d}{d\tilde{a}} \frac{\partial v(a,z,\boldsymbol{g})}{\partial g}[\tilde{a},\tilde{z}]}_{=\text{Lions}} \underbrace{d\tilde{a}}_{=\text{change in decision}} g(\tilde{a},\tilde{z})$$

- $\frac{\partial v(a,z,\boldsymbol{g})}{\partial g}[\tilde{x}]$ Fréchet Derivative, for a change of g in $\tilde{x}$
- $\frac{dv(a,z,\boldsymbol{g})}{dg}[\tilde{x}] = \frac{d}{dx}\frac{\partial v(a,z,\boldsymbol{g})}{\partial g}[\tilde{x}]$ Lions Derivative, for a change of $\tilde{x}$, i.e. a *shift* in $g(\tilde{x})$





back

Agents' decision and global dynamical system

- With $v = \bar{v}(a, z, K, Z)$, we get individual decisions:

$$c^\star(\tilde{a}, \tilde{z}, K, Z) = \begin{cases} u'^{-1}\big(\bar{v}_a(\tilde{a}, \tilde{z}, K, Z)\big) \\ w(K,Z)z + r(K,Z)a \end{cases}$$

$\Rightarrow$ Dynamical system for $x = (a, z, K, Z)$

$$\begin{cases} da &= \Big[z\overbrace{(1-\alpha)Z\,K^\alpha}^{=w(K,Z)} + \overbrace{(\alpha Z\,K^{\alpha-1} - \delta)}^{=r(K,Z)}\,a - c^\star(a, z, K, Z)\Big]dt \\ dz &= \gamma(z)dJ_t \qquad \text{Markov, w/ intensity} \qquad \lambda(z) \\ dK &= \big(ZK^\alpha - \delta K - \mathcal{C}(K, Z|g)\big)dt \\ dZ &= -\theta(Z - \bar{Z})dt + \widehat{\sigma}dB_t^0 \end{cases}$$

- For a guess of $g_{(a,z)}$ and $\mathcal{C}(K, Z|g)$ $\Rightarrow$ complete characterization

$\Rightarrow$ Obtain a Kolmogorov Forward equation for system (a, z, K, Z) (!)

back

"Master-" Kolmogorov Forward for the global system

- For a guess of $g(a,z)$ and $\mathcal{C}(K,Z|g) = \iint_{a,z} c^{\star}(a,z,K,Z)\, dg(a,z)$
  - We can solve a "Master-KFE" for states $x=(a,z,K,Z)\in\mathbb{X}$ to find the distribution $\widetilde{g}(x)$

$$0 = -\partial_a\left[s(x,\bar{v}_a)\widetilde{g}(x)\right] + \sum_n \lambda(z^n)\widetilde{g}(x^n) - \lambda(z)\widetilde{g}(x)$$

$$-\partial_K\left[\left(ZK^{\alpha} - \delta K - \mathcal{C}(K,Z|g)\right)\widetilde{g}(\widetilde{x})\right] - \partial_Z[-\theta(Z-\bar{Z})\widetilde{g}(x)] + \widehat{\sigma}\partial^2_{ZZ}\widetilde{g}(x)$$

  - Easy to get $\widetilde{g}$ from Master-HJB's operator $\mathcal{A}[\bar{v}]$ with finite-difference methods

back