

The Master Equation, Projection, and Aggregation for Heterogeneous Agents models with Aggregate Risk

WORK IN PROGRESS

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Inequality, Macro and Uncertainty

- ▶ Inequality matters for **macro** and **aggregate risk**, and they matter for Inequality
 - Recessions hit households unequally, and consumption, saving and portfolio choices respond to aggregate **uncertainty**.
- ▶ Main **difficulty** introducing **risk** in models with **inequality/heterogeneity**
 - With rational expectations, agents need to forecast price/aggregate dynamics
 - ⇒ The distribution of agents g enters the household/firm decision problem
 - It gives rise to a “Master equation”, the value depends on an infinite-dimensional object g
- ▶ I provide a method to tackle and simplify this Master Equation
 - Merge Krusell-Smith’s original idea of **projection on moments** with modern math treatments
 - ▷ Provide applications with meaningful risk and non-linearity.

This project

- ▶ I suggest a new method to solve HA models with **aggregate risk**
 - State-of-the-art / First-Order methods (e.g. SSJ) rely mostly on **certainty equivalence**
 - More recent global methods (e.g. Deep Learning) may seem a little opaque or case-specific
- ▶ I go back to the original idea of Krusell-Smith (1998)
 - I (still) summarize the distribution of agents with a finite set of moments: **“projection”**
- ▶ But I borrow from the mathematics literature: **Master equation**
 - MFGs w/ Master equation: *Cardaliaguet, Delarue, Lions, Lasry (2019) Carmona, Delarue (2018) + many others*
 - Also introduced in economics: *Schaab (2021), Bilal (2023), Gu, Laurière, Merkel, Payne (2024)*
- ▶ Can get additional **insights** on the nature of the equilibrium:
 - What are the drivers and amplification channels of aggregate fluctuations?
 - Are those restricted beliefs a good approximation for aggregation?
 - Applications with non-linearity, aggregate uncertainty, or oligopoly.

Outline

1. Krusell Smith Model
2. Primer on the Master equation
3. Projection in Heterogeneous Agents Models
4. Numerical results for KS98
5. Testing bounded-rationality assumption in KS98.
6. “Macrofinance”: portfolio choice and Second Order Master Equation
7. Dynamic Oligopoly: industry dynamics and Discrete Master Equation

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Krusell-Smith (KS98) recap

- Consumption-saving model, c, a , with (i) idiosyncratic income risk z , (ii) incomplete market (iii) credit constraints $a \geq \underline{a}$, (iv) aggregate shock on aggregate TFP Z
- HH's distribution $g(a, z)$ over wealth a and income z
- Firm side:

$$Y = ZK^\alpha \quad \Rightarrow \quad r = \alpha ZK^{\alpha-1} - \delta \quad w = (1-\alpha)K^\alpha$$

- Household problem (KS98)

$$V(a, z, g, Z) = \max_{c, a'} u(c) + \beta \mathbb{E}^{z', Z'} [V(a', z', g', Z') \mid z, Z]$$

$$s.t. \quad c + a' = zw + (1+r)a$$

$$g' = H(g, Z, Z')$$

- Equilibrium

$$K = \int_{a, z} a g(da, z) \quad \forall Z$$

General idea and KS98 global solution

► Difficulty:

- Value $V(a, z, g, Z)$ depends on the whole distribution g
- Need to forecast the evolution of g , possibly on *every path* of $\{Z_t\}_t$, Achdou, Bourany (2018)

► Krusell-Smith solution

1. Assume Households only care about agg. capital / First-moment $K = \int a g(da, z)$
2. Assume *Linear* forecasting rule:

$$\log K' = a_1^Z \log K + a_2^Z$$

- Choose parameters (a_1^Z, a_2^Z) to match the *realized / simulated* path (Monte Carlo) of $\{K_t\}_t$

► Proposal today: using the Master equation to:

- remove assumption 2 $\Rightarrow$ bypass the linearity assumption $K' = b(K, Z)$
- build macro dynamics $b(K, Z)$ from the *aggregation of micro* decisions
- test robustness to 1 and 2, by adding more moments

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Primer on Mean Field Games (MFG): the Aiyagari model

- States dynamics: saving and income shocks

$$da_t = [z_t w_t + r_t a_t - c_t] dt \quad z_j \sim \text{Markov jump process } \lambda_j$$

1. Hamilton Jacobi Bellman Equation:

$$-\partial_t v(t, a, z) + \rho v(t, a, z) = \max_c u(c) + \mathcal{L}[v | c](t, a, z)$$

- Transport/Jump-Operator $\mathcal{L}$: *from agents' decision and shocks*

$$\mathcal{L}[v | c^*](t, a, z_j) = \underbrace{\partial_a v(t, a, z_j) [z_j w + r a - c^*]}_{\text{change in saving}} + \underbrace{\lambda_j (v(t, a, z_{-j}) - v(t, a, z_j))}_{\text{change in labor income}}$$

2. Kolmogorov Forward Equation:

$$\partial_t g(t, a, z) = \mathcal{L}^*[g | c^*](t, a, z)$$

– $\mathcal{L}^*$: *distribution dynamics comes from agents' decisions*

- General Equilibrium, coupling through r, w :

$$\iint_{z, a \geq \underline{a}} a g(t, da, z_j) = K \quad r = \alpha \bar{Z} K^{\alpha-1} - \delta$$

Primer on the Master Equation

► Master equation combines both **HJB** and **KFE** in *one equation*

- No aggregate risk

$$-\partial_t v(t, a, z, \mathbf{g}) + \rho v(t, a, z, \mathbf{g}) = \overbrace{\max_c u(c) + \mathcal{L}[v | c^*](t, a, z)}^{\text{standard HJB continuation value}} +$$

$$\underbrace{\int_{z, a} \frac{\delta v(t, a, z, \mathbf{g})}{\delta \mathbf{g}}[\tilde{a}, \tilde{z}] \mathcal{L}^*[\mathbf{g} | c^*](t, \tilde{a}, \tilde{z})}_{\text{change in } v \text{ due to the distribution dynamics}}$$

- First part: **HJB**, how states (a, z) change agents' value v
- **Novelty**: how the distribution $\mathbf{g}$ changes the value v
 - Forecast of agents (a, z) about *all* the other agents $(\tilde{a}, \tilde{z})$
 - 1. $\mathcal{L}^*[\mathbf{g} | c^*]$: Agents' decisions change the distribution $\mathbf{g}$
 - 2. $\frac{\delta v}{\delta \mathbf{g}}[\tilde{x}]$: How the distribution changes the value, when adding mass at $\tilde{x}$

Adding Aggregate Risk to the Master Equation

► Consider aggregate risk

- Agg. TFP follows a AR(1) – Ornstein-Uhlenbeck process

$$dZ_t = -\theta(Z_t - \bar{Z})dt + \hat{\sigma} dB_t^0$$

- Value $v = v(t, a, z, g, Z)$

$$\begin{aligned} -\partial_t v + \rho v = & \underbrace{\max_c u(c) + \mathcal{L}[v|c](t, a, z)}_{\text{standard HJB continuation value}} \quad \underbrace{-\theta(Z - \bar{Z})v_Z + \frac{\hat{\sigma}^2}{2} v_{ZZ}}_{\text{direct effect of risk of } Z \text{ on } v} \\ & + \underbrace{\iint_{z, a} \frac{\delta v(t, a, z, g, Z)}{\delta g} [\tilde{a}, \tilde{z}] \mathcal{L}^*[g|c^*](t, \tilde{a}, \tilde{z})}_{\text{change due to distribution dynamics}} \end{aligned}$$

- Master equation doesn't change much: Why?

- No *direct effects* on individual states from agg. shocks: distribution g is not affected by dB_t^0
- If it were (portfolio problems, see later): becomes *second order* $\Rightarrow$ Monster equation!

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Projection assumption in Krusell-Smith (KS98)

- What do households need for decisions? *only* changes in prices (r, w)
 $\Rightarrow$ Do not care of distribution g *per se*, Only one moment, K , *the mean* of g to forecast prices

$$K = \iint_{a,z} a g(da, z) \qquad r = \alpha Z K^{\alpha-1} - \delta$$

- **Assumption:** Restricted belief (here as in KS98)

$$v(a, z, g, Z) = \bar{v}(a, z, K^h, Z) + \varepsilon$$

– Dimensionality reduction: $\mathbb{E}[\varepsilon \mid g \text{ s.t. } K = \iint a g(da, z)] = 0 \quad \Rightarrow \quad \text{information loss}$

- **Novelty:** Nice property in Lions-derivative for the Master equation

$$\text{with } K^h = \int_x h(x) g(dx) \qquad \frac{\delta v(x, g)}{\delta g}[\tilde{x}] \equiv \frac{d\bar{v}(x, K^h)}{dK^h} h(\tilde{x})$$

Projection in the Master equation

- With first-moment: $v(a, z, \mathbf{g}, \mathbf{Z}) \approx \bar{v}(a, z, \mathbf{K}, \mathbf{Z})$

$$\rho \bar{v} = \overbrace{\max_c u(c) + \mathcal{L}[\bar{v} | c]_{(a, z, \mathbf{K}, \mathbf{Z})}}^{\text{standard HJB continuation value}} \overbrace{-\theta(Z - \bar{Z})\bar{v}_Z + \frac{\hat{\sigma}^2}{2}\bar{v}_{ZZ}}^{\text{direct effect of risk of } \mathbf{Z} \text{ on } \bar{v}} + \bar{v}_K \iint_{z, a} \underbrace{[r\tilde{a} + w\tilde{z} - c^*(\tilde{a}, \tilde{z}, \mathbf{K}, \mathbf{Z})]}_{\text{change in agents } (\tilde{a}, \tilde{z}) \text{ decisions}} g(d\tilde{a}, \tilde{z})$$

- Aggregation:

$$dK = \iint_{z, a} [r\tilde{a} + w\tilde{z} - c^*(\tilde{a}, \tilde{z}, \mathbf{K}, \mathbf{Z})] g(d\tilde{a}, \tilde{z})$$

$$d\mathbf{K} = r\mathbf{K} + w\bar{L} - \mathcal{C}(\mathbf{K}, \mathbf{Z} | g) \quad \text{w/ agg conso: } \mathcal{C}(\mathbf{K}, \mathbf{Z} | g) = \iint_{z, a} c^*(\tilde{a}, \tilde{z}, \mathbf{K}, \mathbf{Z}) g(d\tilde{a}, \tilde{z})$$

- Micro-foundation of the law of motion $d\mathbf{K}$ through aggregation
- But still dependence on g ! How to “get rid of it”?

Master Equation becomes a fusion of two familiar HJB equations

► With $v = v(a, z, \mathbf{g}, \mathbf{Z}) \approx \bar{v}(a, z, K, \mathbf{Z})$

$$\begin{aligned} \rho \bar{v} = \max_c u(c) + [wz + ra - c] \bar{v}_a + \lambda(\bar{v}(a, z', \cdot) - \bar{v}(a, z, \cdot)) \\ - \theta(Z - \bar{Z}) \bar{v}_Z + \frac{\hat{\sigma}^2}{2} \bar{v}_{ZZ} + \underbrace{[ZK^\alpha - \delta K - \mathcal{C}(K, \mathbf{Z} | \mathbf{g})]}_{=dK} \bar{v}_K \end{aligned}$$

• Looks like a fusion of two standard models

– RBC: $v = v(K, Z)$

$$\rho v = \max_C u(C) + [ZK^\alpha - \delta K - C]v_K - \theta(Z - \bar{Z})v_Z + \frac{\hat{\sigma}^2}{2} v_{ZZ}$$

– Aiyagari: $v = v(a, z)$

$$\rho v = \max_c u(c) + [wz + ra - c]v_a + \lambda(v(a, z', \cdot) - v(a, z, \cdot))$$

Agents' decision and global dynamical system

- With $v = \bar{v}(a, z, K, Z)$, we get individual decisions:

$$c^*(\tilde{a}, \tilde{z}, K, Z) = \begin{cases} u'^{-1}(\bar{v}_a(\tilde{a}, \tilde{z}, K, Z)) \\ w(K, Z)z + r(K, Z)a \end{cases}$$

- Thus, get a dynamical system for $x = (a, z, K, Z)$
- Kolmogorov Forward equation for $x \Rightarrow$ Gives a distribution $\tilde{g}(x)$
- Easy to get $\tilde{g}$ from Master Eq. operator $\mathcal{A}[\bar{v}|c^*] \Leftrightarrow \mathcal{A}^*[\tilde{g}|c^*]$

Dynamical system

"Master" KFE

- Consistency condition for rational-expectation equilibrium: $\sim$ Bayes rules / Radon-Nykodim

$$g(da, z_j)|_{K, Z} = g(da, z_j|K, Z) = \frac{\tilde{g}(da, z_j, dK, dZ)}{\int_{\mathbb{X}} \tilde{g}(da, z_j, dK, dZ)}$$

- Using this g get $\mathcal{C}(K, Z|g) \Rightarrow$ all we needed!!

Summary and numerical methods

1. General Master equation for $v(a, z, g, Z)$

2. Master Equation with “projection”: $v = \bar{v}(a, z, K, Z)$

- Start from guess $g(a, z)$, $\mathcal{C}(K, Z|g)$, and $dK^{(n)}$
- Solve Master Equation (4 dims): finite difference methods
- Get ind. decisions $c^*(a, z, K, Z)$ and operator $\mathcal{A}[\bar{v}]$

3. “Master”-Kolmogorov forward for $x = (a, z, K, Z)$

- Solve for distribution $\tilde{g}$ “for free” with $\mathcal{A}^*[\tilde{g}]$
- Update g thanks to $\tilde{g} \rightarrow$ update $\mathcal{C}(K, Z|g)$
- Update agg. dynamics until convergence:

$$dK^{(n+1)} = ZK^\alpha - \delta K - \mathcal{C}(K, Z|g)$$

► General procedure

- No need for deep-learning/splines/polynomials: use “**standard**” finite difference methods
- **Aggregation** from micro-decisions $\rightarrow$ potentially very non-linear!
- **Rational Expectation**: Does **not** rely on bounded-rationality assumption of KS98

Master-Equation with higher moments:

- Extension with second moments: $v(a, z, \mathbf{g}, Z) \approx \bar{v}(a, z, K, K_2, KL, \mathbf{Z})$

- $K_2 = \mathbb{V}\text{ar}(a)$, $KL = \mathbb{C}\text{ov}(a, z)$.

$$\begin{aligned} \rho \bar{v} = \max_c u(c) + \mathcal{L}[\bar{v} | c] &- \theta(Z - \bar{Z})\bar{v}_Z + \frac{\hat{\sigma}^2}{2}\bar{v}_{ZZ} + \underbrace{\left(\mathbf{Z}K^\alpha - \delta K - \mathbb{E}^g[c^*] \right)}_{=dK} \bar{v}_K \\ &+ \underbrace{\mathbb{C}\text{ov}^g(a, s^*)}_{dK_2} \bar{v}_{K_2} - (\lambda_1 + \lambda_2) \underbrace{\mathbb{C}\text{ov}^g(a, z)}_{dKL \propto -KL} \bar{v}_{KL} \end{aligned}$$

- System: $x = (a, z, K, K_2, KL, \mathbf{Z})$: $\bar{v}(x) \Rightarrow c^*(x)$ and $dK = \mathcal{S}(K, K_2, KL, \mathbf{Z} | g)dt$
- Theoretical insights: (numerics WIP)
 - With more moments (= information), one predict more accurately price dynamics (r, w)
 - Distribution over moments $\tilde{g}(x)$: self-regulating forces (e.g. KL)
 - If $\mathbb{C}\text{ov}^g(a, s^*) > 0 \Rightarrow K_2 \uparrow$, and if $\bar{v}_{K_2} < 0$, $\rightarrow$ reinforces precautionary saving

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Numerical results – Summary

► Risk:

- Idiosyncratic risk: Two-state Markov process for labor income shocks z
- Aggregate risk: Three-state Markov process for TFP Z , with $\sigma(Z) = 12\%$

► I compare three economies:

1. RBC (Brock Mirman) model

⇒ (v, g) value and distribution over aggregate capital and TFP (K, Z) .

2. Aiyagari model

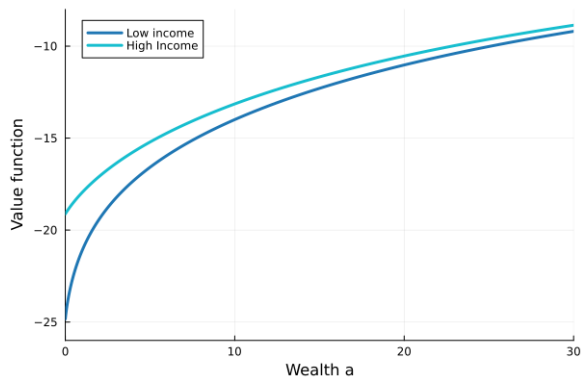
⇒ (v, g) individual heterogeneity on (a, z) , for constant TFP $Z = \bar{Z}$ and capital $K = \bar{K}$.

3. Krusell-Smith model

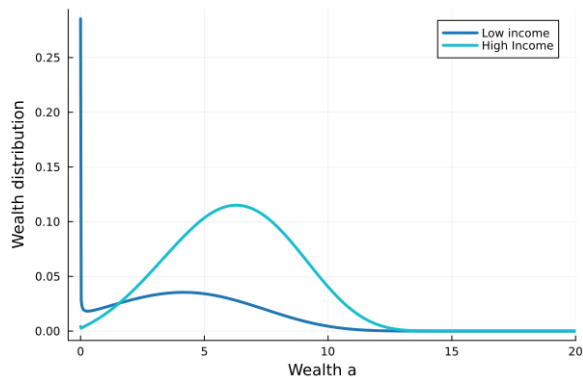
- Have both: (v, g) over (a, z, K, Z)
- Iterate over $dK = ZK^\alpha - \delta K - C(K, Z|g)$

Recap – Aiyagari model

Value function $v(a, z)$

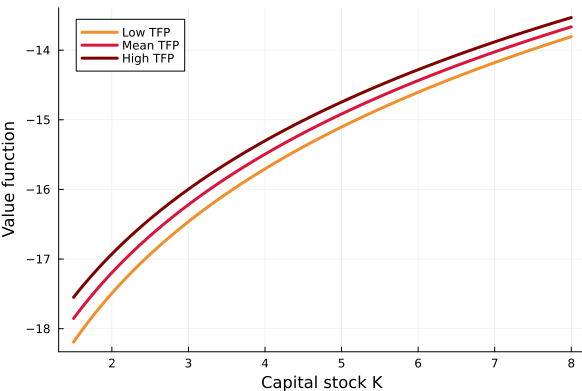


Distribution $g(a, z)$

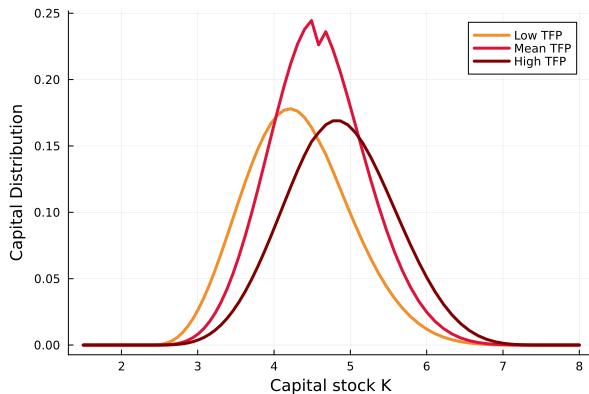


Recap – Brock-Mirman / RBC

Value function $v(K, Z)$

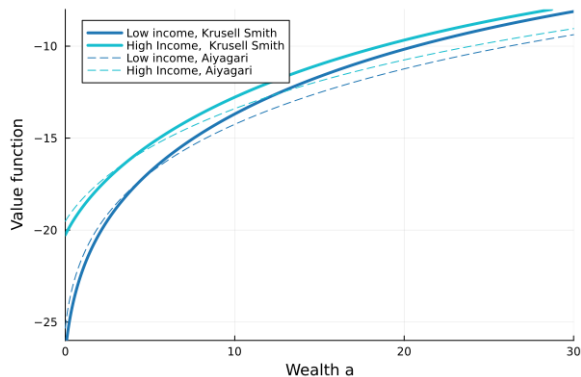


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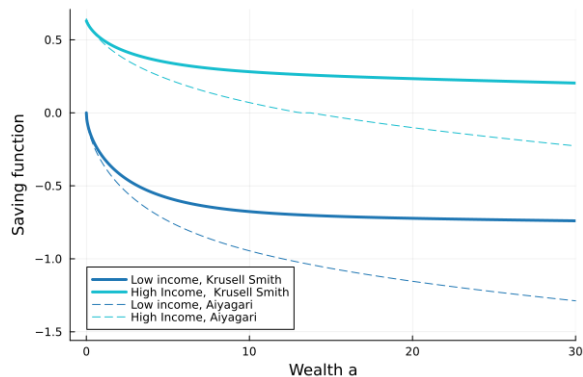


Krusell-Smith: individual decisions, vs. Aiyagari

Value function $v(a, z, \bar{K}, \bar{Z})$

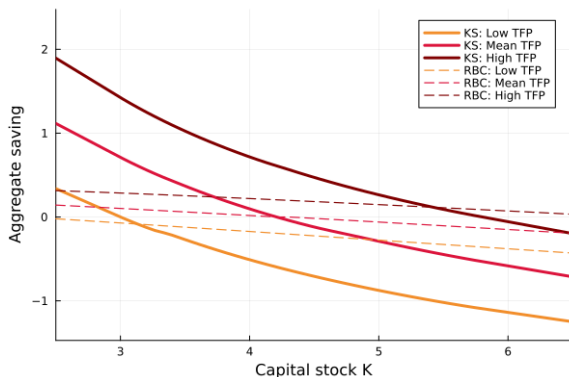


Saving $s(a, z, \bar{K}, \bar{Z}) = wz + ra - c^*$

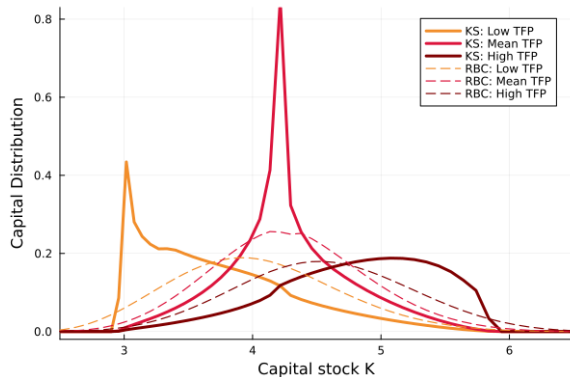


Krusell-Smith: aggregate dynamics decisions, vs. RBC

Aggregate dynamics $dK = \textcolor{blue}{Z}K^\alpha - \delta K - \mathcal{C}(K, \textcolor{blue}{Z}|g)$



Distribution $\tilde{g}(K, \textcolor{blue}{Z})$



Effect of aggregate uncertainty and heterogeneity

1. Higher individual precautionary saving motives:
 - Aggregate uncertainty affects the poor (reliant on labor income) more than the rich (hedging: $r = MPK \uparrow$ when $K \downarrow$)
 - Implies much more savings when rich: flatter savings function compared to Aiyagari
2. Aggregate dynamics are more “reactive”
 - Individual heterogeneity implies a steeper aggregate saving function
 - Rich household save a lot more when K is lower, boosting investment
3. Change the distribution of economic activity
 - Output is fluctuating more in Krusell-Smith: 15.5% volatility compared to 14% for RBC
 - However, business cycles are less skewed – smaller right tail, more symmetric – and have less kurtosis – thinner tails of capital/output.

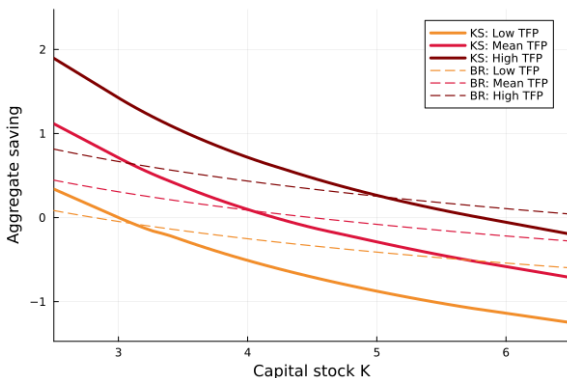
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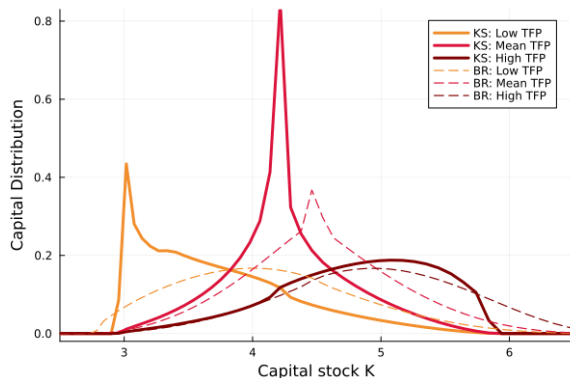
Bounded-rationality in Krusell-Smith

- Agents do not forecast the “true” capital dynamics dK
 - Assume a log-linear rule: $dK = \beta^Z \log(K) dt$
 - What do we “lose” from this assumption?

Aggregate dynamics $dK = \beta^Z \log(K) dt$



Distribution $\tilde{g}(K, Z)$



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Master equation and Merton portfolio problem

- Agg. risk: both TFP (OU) $Z \sim \widehat{\sigma} dB_t^0$ and *direct effect* on portfolio return (share α)

$$da = (ra + zw - c)dt + \alpha a (r^k - r)dt + \alpha a \overline{\sigma} dB_t^0$$

- The master equation now becomes **second order**! value $v = v(a, z, g, Z)$ changes a lot!

$$\begin{aligned} \rho v = & \underbrace{\max_{c, \alpha} u(c) + \mathcal{L}[v|c, \alpha](a, z, g, Z)}_{\text{standard HJB continuation value}} \underbrace{-\theta(Z - \bar{Z})v_Z + \frac{\widehat{\sigma}^2}{2} v_{ZZ}}_{\text{direct effect of risk of } Z \text{ on } v} + \underbrace{\iint_{z, a} \frac{dv(a, z, g, Z)}{dg} [\tilde{a}, \tilde{z}] \mathcal{L}^*[g|c^*](\tilde{a}, \tilde{z})}_{\text{deterministic evolution of the distribution}} \\ & + \underbrace{\frac{\overline{\sigma}^2}{2} \iint_{z, a} \alpha(\tilde{a}, \tilde{z})^2 \tilde{a}^2 \frac{d}{d\tilde{a}} \left(\frac{dv}{dg} [\tilde{a}, \tilde{z}] \right) g(d\tilde{a}, \tilde{z})}_{\text{diffusion of the distribution due to risk}} + \underbrace{\overline{\sigma}^2 \alpha(a, z) \iint_{z, a} \alpha(\tilde{a}, \tilde{z}) \tilde{a} \frac{d}{da} \frac{dv}{dg} [\tilde{a}, \tilde{z}] g(d\tilde{a}, \tilde{z})}_{\text{covariance of own state } a \text{ and distribution } \tilde{a}} \\ & + \underbrace{\overline{\sigma} \widehat{\sigma} \iint_{z, a} \alpha(\tilde{a}, \tilde{z}) \frac{d}{dZ} \frac{dv}{dg} [\tilde{a}, \tilde{z}] g(d\tilde{a}, \tilde{z})}_{\text{covariance of agg. state } Z \text{ and distribution } \tilde{a}} + \underbrace{\frac{\overline{\sigma}^2}{2} \iint_{(z, a)^{\otimes 2}} \alpha(\tilde{a}, \tilde{z}) \alpha(\tilde{a}', \tilde{z}') \frac{d^2 \bar{v}}{dg^2} [\tilde{a}, \tilde{z}, \tilde{a}', \tilde{z}'] g(d\tilde{a}, \tilde{z}) g(d\tilde{a}', \tilde{z}')}_{\text{covariance of distribution } \tilde{a} \text{ and } \tilde{a}'} \end{aligned}$$

Master equation with projection for portfolio problem

- Agg. risk: both TFP (OU) $Z \sim \hat{\sigma} dB_t^0$ and *direct effect* on portfolio return (share α)
- With projection $v(a, z, g, Z) = \bar{v}(a, z, K, Z)$, master equation simplifies:

$$\begin{aligned}
 \rho \bar{v} = & \underbrace{\max_{c, \alpha} u(c) + \mathcal{L}[\bar{v}|c, \alpha](a, z, K, Z)}_{\text{standard HJB continuation value}} \underbrace{-\theta(Z - \bar{Z})\bar{v}_Z + \frac{\hat{\sigma}^2}{2}\bar{v}_{ZZ}}_{\text{direct effect of risk of } Z \text{ on } v} + \underbrace{\left(ZK^\alpha - \delta K - \mathcal{C}(K, Z|g) \right) \bar{v}_K}_{\text{deterministic evolution of the distribution}} \\
 & + \underbrace{\frac{\bar{\sigma}^2}{2} K \bar{v}_K \times 0}_{\text{diffusion of distribution due to risk} = 0} + \underbrace{\frac{\bar{\sigma}^2}{2} \alpha(a, z, K, Z) a K \bar{v}_{Ka}}_{\text{covariance of own state } a \text{ and moment } K} \\
 & + \underbrace{\frac{\bar{\sigma} \hat{\sigma}}{2} K \bar{v}_{KZ}}_{\text{covar. of agg. state } Z \text{ and moment } K} + \underbrace{\frac{\bar{\sigma}^2}{2} K^2 \bar{v}_{KK}}_{\text{Var moment } K} \quad \text{with } K = A(K, Z|g) = \iint_{z, a} \alpha(\tilde{a}, \tilde{z}, \cdot) \tilde{a} g(d\tilde{a}, \tilde{z})
 \end{aligned}$$

Optimal decisions and portfolio

- Agg. risk: both TFP (OU) $Z \sim \hat{\sigma} dB_t^0$ and *direct effect* on portfolio return (share α)

- With projection $v(a, z, g, Z) = \bar{v}(a, z, K, Z)$
- Optimal decisions: consumption

$$c^*(a, z_j, K, Z) = \begin{cases} u'^{-1}(\bar{v}_a(a, z_j, K, Z)) & a > \underline{a} \\ z_j w(K, Z) + r a & \text{if } a \leq \underline{a} \end{cases}$$

- Optimal portfolio choice:

$$\alpha^*(a, z_j, K, Z) = \frac{r^k - r}{\gamma \bar{\sigma}^2} + \frac{\hat{\sigma}}{\bar{\sigma} \gamma} \frac{\bar{v}_{aZ}}{\bar{v}_a} + \frac{K \bar{v}_{aK}}{\gamma \bar{v}_a} \quad \text{if } a > \underline{a}$$

and $\alpha^*(a, z_j, K, Z) = 0$ if $a \leq \underline{a}$ with $\frac{1}{\gamma(x)} = -\frac{\bar{V}_a(a, z_j, K, Z)}{\bar{V}_{aa}(a, z_j, K, Z)a}$

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1. Krusell Smith Model
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3. Projection in Heterogeneous Agents Models
4. Numerical results for KS98
5. Testing bounded-rationality assumption in KS98.
6. “Macrofinance”: portfolio choice and Second Order Master Equation
7. **Dynamic Oligopoly: industry dynamics and Discrete Master Equation**

Dynamic Oligopoly – Ericson-Pakes

- ▶ Firms i with cost state $x \in \mathbb{X}$, investment a with inv. cost $c(x)a$
 - x drift $\mu(x, \mathbf{Z}, a)$, jump up w/ proba $\phi(x, \mathbf{Z}, a)$, fall w/ proba δ + entry / exit
- ▶ Market state $\mathbf{x} = \{x_i\}_i \Leftrightarrow$ emp. distribution of firms $\mathbf{g} = \sum_i \delta_{x_i}$.
- ▶ Aggregate cost $\mathbf{Z}$, following AR(1)-OU process $\hat{\sigma}$
- ▶ Competition à la Bertrand, CES pref. σ
- ▶ Market share $s_i = \left(\frac{p_i}{p}\right)^{1-\sigma}$, and elasticity $\varepsilon_i(s_i) = \sigma - (\sigma - 1)s_i$
- $\Rightarrow$ Firm profit $\pi(x_i, \mathbf{x}, \mathbf{Z}) = \frac{Es_i}{\sigma - (\sigma - 1)s_i} - f$
- ▶ Static optimal pricing $p_i = \frac{\varepsilon_i}{\varepsilon_i - 1} x_i \mathbf{Z}$
- ▶ Interaction in oligopolistic markets: price index $P^{1-\sigma} = \sum_i p_i^{1-\sigma}$

Dynamic Oligopoly – Master Equation

- Master equation: $v = v(x_i, g, Z)$, with a discrete distribution / finite states

$$\rho v = \overbrace{\max_a \pi(x_i, g, Z) - c(x_i) a + \mathcal{L}[v|a]}^{\text{=standard HJB cont. value}} \overbrace{-\theta(Z - \bar{Z}) \bar{v}_Z + \frac{\hat{\sigma}^2}{2} v_{ZZ}}^{\text{=direct effect of agg. risk } Z} + \overbrace{\int_x \frac{\delta v}{\delta g}[\tilde{x}] \mathcal{L}^*[g|a^*](d\tilde{x})}^{\text{=evolution of the distribution}}$$

- Operator $\mathcal{L}[v|a] = \mu(x, Z, a) v_x + \phi(x, Z, a) (v_{(x_{i+1}, \cdot)} - v_{(x_i, \cdot)}) + \delta (v_{(x_{i-1}, \cdot)} - v_{(x_i, \cdot)})$

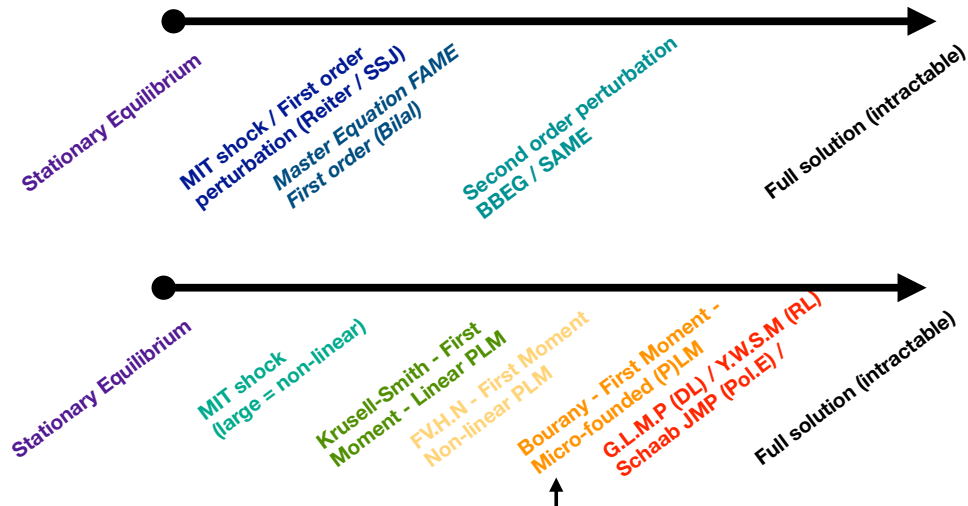
- Payoff-relevant agg. state: P . Projection $v(x_i, g, Z) \approx v(x_i, P, Z)$

$$\rho \bar{v} = \overbrace{\max_a \pi(x_i, P, Z) - c(x_i) a + \mathcal{L}[\bar{v}|a]}^{\text{=standard HJB cont. value}} \overbrace{-\theta(Z - \bar{Z}) \bar{v}_Z + \frac{\hat{\sigma}^2}{2} \bar{v}_{ZZ}}^{\text{=direct effect of agg. risk } Z} + \bar{v}_P \overbrace{\int_x \left(\frac{p(x_j, P)}{P} \right)^{-\sigma} \mu(x_i, Z, a^*) g(dx)}^{\text{=Cov}^g \left(s_i^{\frac{\sigma}{\sigma-1}}, \mu(x, a) \right)}$$

$$+ \sum_j \hat{\Phi}_j^g (\bar{v}_P(\hat{P}^j) - \bar{v}_P) + \sum_j \check{\Phi}_j^g (\bar{v}_P(\check{P}^j) - \bar{v}_P)$$

with $\hat{P}^i = \left(P^{1-\sigma} + \frac{1}{N} (p_{i+1}^{1-\sigma} - p_i^{1-\sigma}) \right)^{\frac{1}{1-\sigma}}$, firm i switch up, with proba $\hat{\Phi}_i^g = \phi(x_i, Z, a_i^*) g(x_i)$

Summary and comparison



Conclusion

- ▶ I propose a new method for solving **Heterogeneous Agent** Models with **Aggregate Risk**
 - Leverage the **Master equation** approach and generalize Krusell-Smith insights with projections on finite sets of moments
- ▶ Next steps:
 - Additional example of information frictions models
 - Macro-finance model to the test + Replicate *Fernandez-Villaverde, Hurtado, Nuno (2023 Ecma)*
 - Oligopoly model a la Ericson Pakes (1995) with a distribution of firms within markets

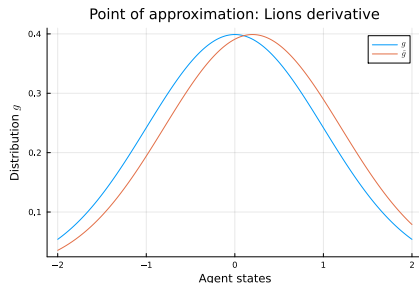
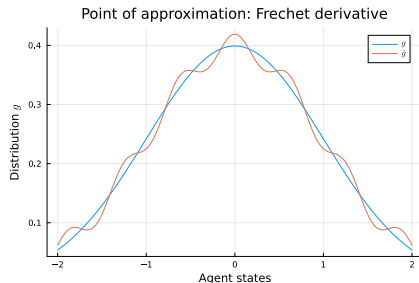
Primer on the Lions derivative

- Derivative in the space of distribution:
how the value $v(a, z, \mathbf{g})$ changes when the distribution of agents $\mathbf{g}$ moves?

$$\begin{aligned} dv(a, z, \mathbf{g}) &\approx v(a, z, \tilde{\mathbf{g}}) - v(a, z, \mathbf{g}) \\ &\approx \iint_{\tilde{a}, \tilde{z}} \underbrace{\frac{\partial v(a, z, \mathbf{g})}{\partial \mathbf{g}}[\tilde{a}, \tilde{z}]}_{=\text{Fréchet}} (\tilde{\mathbf{g}}(\tilde{a}, \tilde{z}) - \mathbf{g}(\tilde{a}, \tilde{z})) \end{aligned}$$

$$\approx \iint_{\tilde{a}, \tilde{z}} \underbrace{\frac{d}{d\tilde{a}} \frac{\partial v(a, z, \mathbf{g})}{\partial \mathbf{g}}[\tilde{a}, \tilde{z}]}_{=\text{Lions}} \underbrace{d\tilde{a}}_{=\text{change in decision}} \mathbf{g}(\tilde{a}, \tilde{z})$$

- $\frac{\partial v(a, z, \mathbf{g})}{\partial \mathbf{g}}[\tilde{x}]$ Fréchet Derivative, for a change of $\mathbf{g}$ in $\tilde{x}$
- $\frac{dv(a, z, \mathbf{g})}{d\mathbf{g}}[\tilde{x}] = \frac{d}{dx} \frac{\partial v(a, z, \mathbf{g})}{\partial \mathbf{g}}[\tilde{x}]$ Lions Derivative, for a change of $\tilde{x}$, i.e. a *shift* in $\mathbf{g}(\tilde{x})$

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Agents' decision and global dynamical system

► With $v = \bar{v}(a, z, K, Z)$, we get individual decisions:

$$c^*(\tilde{a}, \tilde{z}, K, Z) = \begin{cases} u'^{-1}(\bar{v}_a(\tilde{a}, \tilde{z}, K, Z)) \\ w(K, Z)z + r(K, Z)a \end{cases}$$

⇒ Dynamical system for $x = (a, z, K, Z)$

$$\begin{cases} da &= \left[\overbrace{z(1-\alpha)ZK^\alpha}^{=w(K,Z)} + \overbrace{(\alpha ZK^{\alpha-1} - \delta)}^{=r(K,Z)} a - c^*(a, z, K, Z) \right] dt \\ dz &= \gamma(z) dJ_t \quad \text{Markov, w/ intensity} \quad \lambda(z) \\ dK &= (ZK^\alpha - \delta K - \mathcal{C}(K, Z|g)) dt \\ dZ &= -\theta(Z - \bar{Z})dt + \hat{\sigma} dB_t^0 \end{cases}$$

• For a guess of $g_{(a,z)}$ and $\mathcal{C}(K, Z|g)$ ⇒ complete characterization

⇒ Obtain a Kolmogorov Forward equation for system (a, z, K, Z) (!)

“Master-” Kolmogorov Forward for the global system

- For a guess of $g(a, z)$ and $\mathcal{C}(K, Z|g) = \iint_{a,z} c^*(a, z, K, Z) dg(a, z)$
- We can solve a “Master-KFE” for states $x = (a, z, K, Z) \in \mathbb{X}$ to find the distribution $\tilde{g}(x)$

$$0 = -\partial_a \left[s(x, \bar{v}_a) \tilde{g}(x) \right] + \sum_n \lambda(z^n) \tilde{g}(x^n) - \lambda(z) \tilde{g}(x) \\ - \partial_K \left[(ZK^\alpha - \delta K - \mathcal{C}(K, Z|g)) \tilde{g}(\tilde{x}) \right] - \partial_Z [-\theta(Z - \bar{Z}) \tilde{g}(x)] + \hat{\sigma} \partial_{ZZ}^2 \tilde{g}(x)$$

- Easy to get $\tilde{g}$ from Master-HJB’s operator $\mathcal{A}[\bar{v}]$ with finite-difference methods

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