

The Master Equation, Projection, and Aggregation for Heterogeneous Agents models with Aggregate Risk

WORK IN PROGRESS

Thomas Bourany
COLUMBIA / EIEF

April 2025

Solving Heterogeneous Agents models with aggregate risk is hard

- ▶ Main difficulty with introducing risk in models with inequality/heterogeneity
 - With rational expectations (FIRE), agents need to forecast price/aggregate dynamics
- ⇒ The distribution of agents g enters the household/firm decision problem
 - It gives rise to a “Master equation”, the value depends on an infinite-dimensional object g

Solving Heterogeneous Agents models with aggregate risk is hard

- ▶ Main difficulty with introducing risk in models with inequality/heterogeneity
 - With rational expectations (FIRE), agents need to forecast price/aggregate dynamics
 - ⇒ The distribution of agents g enters the household/firm decision problem
 - It gives rise to a “Master equation”, the value depends on an infinite-dimensional object g
- ▶ I provide a method to tackle and simplify this Master Equation
 - Merge Krusell-Smith’s original idea of projection on moments with modern math treatments

Solving Heterogeneous Agents models with aggregate risk is hard

- ▶ Main difficulty with introducing risk in models with inequality/heterogeneity
 - With rational expectations (FIRE), agents need to forecast price/aggregate dynamics
 - ⇒ The distribution of agents g enters the household/firm decision problem
 - It gives rise to a “Master equation”, the value depends on an infinite-dimensional object g

- ▶ I provide a method to tackle and simplify this Master Equation
 - Merge Krusell-Smith’s original idea of projection on moments with modern math treatments

 - ▷ Provide applications with meaningful risk and non-linearity:
 - (i) Benchmark “non-rational expectations” methods
 - (ii) Macro-Finance, asset pricing, and portfolio choice
 - (iii) Dynamic Oligopoly models

This project

- ▶ I suggest a new method (another one?) to solve HA models with **aggregate risk**

This project

- ▶ I suggest a new method (another one?) to solve HA models with **aggregate risk**
 - Huge literature since Krusell-Smith (KS) and Den Haan
 - State of the art – “MIT shocks”, Perturbation methods, *Reiter (2009)*, *Ahn, Kaplan, Moll, Wolf (2018)*, *Bilal (2024)*'s *FAME*, Sequence Space methods *Auclert, Bardóczy, Rognlie, Straub (2021)*, Truncation methods, *Le Grand Ragot (2024)*, and others – rely mostly on **certainty equivalence**

This project

- ▶ I suggest a new method (another one?) to solve HA models with **aggregate risk**
 - Huge literature since Krusell-Smith (KS) and Den Haan
 - State of the art – “MIT shocks”, Perturbation methods, *Reiter (2009)*, *Ahn, Kaplan, Moll, Wolf (2018)*, *Bilal (2024)*'s *FAME*, Sequence Space methods *Auclert, Bardóczy, Rognlie, Straub (2021)*, Truncation methods, *Le Grand Ragot (2024)*, and others – rely mostly on **certainty equivalence**
 - More recent methods bypassing this limitation – Second-order methods, *Bhandari, Bourany, Evans, Golosov (2023)*, *Bayer, Luetticke, Weiss, Winkelmann (2025)*, *Bilal, Franco, Rossi-Hansberg (2026?)*'s *SAME*, Machine-Learning-based methods, *Fernandez-Villaverde, Hurtado, Nuno (2023)*, *Huang (2023)*, *Gu, Laurière, Merkel, Payne (2024)*, *Yang, Wang, Schaab, Moll (2026)*, or others methods *Proehl (2019)*, *Schaab (2021)*, *Lee (2026)* – may seem **case-specific**, algebra heavy, or opaque

This project

- ▶ I suggest a new method to solve HA models with **aggregate risk**
 - State of the art methods rely mostly on **certainty equivalence**
 - More recent methods may seem a little opaque or case-specific

This project

- ▶ I suggest a new method to solve HA models with **aggregate risk**
 - State of the art methods rely mostly on **certainty equivalence**
 - More recent methods may seem a little opaque or case-specific
- ▶ I go back to the original idea of Krusell-Smith (1998)
 - I (still) summarize the distribution of agents with a finite set of moments: **“projection”**

This project

- ▶ I suggest a new method to solve HA models with **aggregate risk**
 - State of the art methods rely mostly on **certainty equivalence**
 - More recent methods may seem a little opaque or case-specific
- ▶ I go back to the original idea of Krusell-Smith (1998)
 - I (still) summarize the distribution of agents with a finite set of moments: **“projection”**
- ▶ But I borrow from the mathematics literature: **Master equation**
 - Mean Field Games and Master equation: *Cardaliaguet, Delarue, Lions, Lasry (2019)*
 - Also introduced in economics: *Schaab (2021), Bilal (2023), Gu, Laurière, Merkel, Payne (2024)*

This project

- ▶ I suggest a new method to solve HA models with **aggregate risk**
 - State of the art methods rely mostly on **certainty equivalence**
 - More recent methods may seem a little opaque or case-specific
- ▶ I go back to the original idea of Krusell-Smith (1998)
 - I (still) summarize the distribution of agents with a finite set of moments: **“projection”**
- ▶ But I borrow from the mathematics literature: **Master equation**
 - Mean Field Games and Master equation: *Cardaliaguet, Delarue, Lions, Lasry (2019)*
 - Also introduced in economics: *Schaab (2021), Bilal (2023), Gu, Laurière, Merkel, Payne (2024)*
- ▶ Can get additional **insights** on the nature of the equilibrium:
 - What are the drivers and amplification channels of aggregate fluctuations?
 - Are those restricted beliefs a good approximation for aggregation?
 - Applications with non-linearity and risk

Outline

1. Krusell Smith Model
2. Primer on the Master equation
3. Projection in Heterogeneous Agents Models
4. Numerical results for KS98
5. Testing bounded-rationality assumption in KS98.
6. “Macrofinance”: portfolio choice and Second Order Master Equation

Outline

1. **Krusell Smith Model**
2. Primer on the Master equation
3. Projection in Heterogeneous Agents Models
4. Numerical results for KS98
5. Testing bounded-rationality assumption in KS98.
6. “Macrofinance”: portfolio choice and Second Order Master Equation

Krusell-Smith (KS98) recap

- Consumption-saving model, c, a , with (i) idiosyncratic income risk z , (ii) incomplete market (iii) credit constraints $a \geq \underline{a}$, (iv) aggregate shock on aggregate TFP Z
- HH's distribution $g(a, z)$ over wealth a and income z

Krusell-Smith (KS98) recap

- Consumption-saving model, c, a , with (i) idiosyncratic income risk z , (ii) incomplete market (iii) credit constraints $a \geq \underline{a}$, (iv) aggregate shock on aggregate TFP Z
- HH's distribution $g(a, z)$ over wealth a and income z
- Firm side:

$$Y = ZK^\alpha \quad \Rightarrow \quad r = \alpha K^{\alpha-1} - \delta \quad w = (1-\alpha)K^\alpha$$

- Household problem (KS98)

$$V(a, z, g, Z) = \max_{c, a'} u(c) + \beta \mathbb{E}^{z', Z'} [V(a', z', g', Z') \mid z, Z]$$

$$s.t. \quad c + a' = zw + (1+r)a$$

$$g' = H(g, Z, Z')$$

- Equilibrium

$$K = \int_{a, z} a g(da, z) \quad \forall Z$$

General idea and KS98 global solution

► Difficulty:

- Value $V(a, z, g, Z)$ depends on the whole distribution g
- Need to forecast the evolution of g , possibly on *every path* of $\{Z_t\}_t$, Achdou, Bourany (2018)

General idea and KS98 global solution

► Difficulty:

- Value $V(a, z, g, Z)$ depends on the whole distribution g
- Need to forecast the evolution of g , possibly on *every path* of $\{Z_t\}_t$, Achdou, Bourany (2018)

► Krusell-Smith solution

1. Assume Households only care about agg. capital / First-moment $K = \int a g(da, z)$
2. Assume *Linear* forecasting rule:

$$\log K' = a_1^Z \log K + a_2^Z$$

- Choose parameters (a_1^Z, a_2^Z) to match the *realized / simulated* path (Monte Carlo) of $\{K_t\}_t$

General idea and KS98 global solution

► Difficulty:

- Value $V(a, z, g, Z)$ depends on the whole distribution g
- Need to forecast the evolution of g , possibly on *every path* of $\{Z_t\}_t$, Achdou, Bourany (2018)

► Krusell-Smith solution

1. Assume Households only care about agg. capital / First-moment $K = \int a g(da, z)$
2. Assume *Linear* forecasting rule:

$$\log K' = a_1^Z \log K + a_2^Z$$

- Choose parameters (a_1^Z, a_2^Z) to match the *realized / simulated* path (Monte Carlo) of $\{K_t\}_t$

► Proposal today: using the Master equation to:

- remove assumption 2 \Rightarrow bypass the linearity assumption $K' = b(K, Z)$
- build macro dynamics $b(K, Z)$ from the *aggregation of micro* decisions
- test robustness to 1 and 2, by adding more moments

Outline

1. Krusell Smith Model
2. **Primer on the Master equation**
3. Projection in Heterogeneous Agents Models
4. Numerical results for KS98
5. Testing bounded-rationality assumption in KS98.
6. “Macrofinance”: portfolio choice and Second Order Master Equation

Primer on Mean Field Games (MFG): the Aiyagari model

- States dynamics: saving and income shocks

$$da_t = [z_t w_t + r_t a_t - c_t] dt \quad z_j \sim \text{Markov jump process } \lambda_j$$

Primer on Mean Field Games (MFG): the Aiyagari model

- States dynamics: saving and income shocks

$$da_t = [z_t w_t + r_t a_t - c_t] dt \quad z_j \sim \text{Markov jump process } \lambda_j$$

1. Hamilton Jacobi Bellman Equation:

$$-\partial_t v(t, a, z) + \rho v(t, a, z) = \max_c u(c) + \mathcal{L}[v | c](t, a, z)$$

- Transport/Jump-Operator \mathcal{L} : *from agents' decision and shocks*

$$\mathcal{L}[v | c^*](t, a, z_j) = \underbrace{\partial_a v(t, a, z_j) [z_j w + r a - c^*]}_{\text{change in saving}} + \underbrace{\lambda_j (v(t, a, z_{-j}) - v(t, a, z_j))}_{\text{change in labor income}}$$

Primer on Mean Field Games (MFG): the Aiyagari model

- States dynamics: saving and income shocks

$$da_t = [z_t w_t + r_t a_t - c_t] dt \quad z_j \sim \text{Markov jump process } \lambda_j$$

1. Hamilton Jacobi Bellman Equation:

$$-\partial_t v(t, a, z) + \rho v(t, a, z) = \max_c u(c) + \mathcal{L}[v | c](t, a, z)$$

- Transport/Jump-Operator \mathcal{L} : from agents' decision and shocks

$$\mathcal{L}[v | c^*](t, a, z_j) = \underbrace{\partial_a v(t, a, z_j) [z_j w + r a - c^*]}_{\text{change in saving}} + \underbrace{\lambda_j (v(t, a, z_{-j}) - v(t, a, z_j))}_{\text{change in labor income}}$$

2. Kolmogorov Forward Equation:

$$\partial_t g(t, a, z) = \mathcal{L}^*[g | c^*](t, a, z)$$

– \mathcal{L}^* : distribution dynamics comes from agents' decisions

- Equilibrium:

$$\iint_{z, a \geq a} a g(t, da, z_j) = K \quad r = \alpha K^{\alpha-1} - \delta$$

Primer on the Master Equation

► Master equation combines both **HJB** and **KFE** in *one equation*

- No aggregate risk

$$-\partial_t v(t, a, z, \mathbf{g}) + \rho v(t, a, z, \mathbf{g}) = \overbrace{\max_c u(c) + \mathcal{L}[v | c^*](t, a, z)}^{\text{standard HJB continuation value}} + \underbrace{\iint_{z, a} \frac{\delta v(t, a, z, \mathbf{g})}{\delta \mathbf{g}}[\tilde{a}, \tilde{z}] \mathcal{L}^*[g | c^*](t, \tilde{a}, \tilde{z})}_{\text{change in } v \text{ due to the distribution dynamics}}$$

- First part: **HJB**, how states (a, z) change agents' value v
- **Novelty**: how the distribution \mathbf{g} changes the value v
 - Forecast of agents (a, z) about *all* the other agents (\tilde{a}, \tilde{z})

Primer on the Master Equation

► Master equation combines both **HJB** and **KFE** in *one equation*

- No aggregate risk

$$-\partial_t v(t, a, z, \mathbf{g}) + \rho v(t, a, z, \mathbf{g}) = \underbrace{\max_c u(c) + \mathcal{L}[v | c^*](t, a, z)}_{\text{standard HJB continuation value}} + \underbrace{\iint_{z, a} \frac{\delta v(t, a, z, \mathbf{g})}{\delta \mathbf{g}}[\tilde{a}, \tilde{z}] \mathcal{L}^*[g | c^*](t, \tilde{a}, \tilde{z})}_{\text{change in } v \text{ due to the distribution dynamics}}$$

- First part: **HJB**, how states (a, z) change agents' value v
- **Novelty**: how the distribution \mathbf{g} changes the value v
 - Forecast of agents (a, z) about *all* the other agents (\tilde{a}, \tilde{z})
 - 1. $\mathcal{L}^*[g | c^*]$: Agents' decisions change the distribution \mathbf{g}
 - 2. $\frac{\delta v}{\delta \mathbf{g}}[\tilde{x}]$: How the distribution changes the value, when adding mass at \tilde{x}

Adding Aggregate Risk to the Master Equation

- ▶ Consider aggregate risk
 - Agg. TFP follows a AR(1) – Ornstein-Uhlenbeck process

$$dZ_t = -\theta(Z_t - \bar{Z})dt + \hat{\sigma}dB_t^0$$

Adding Aggregate Risk to the Master Equation

► Consider aggregate risk

- Agg. TFP follows a AR(1) – Ornstein-Uhlenbeck process

$$dZ_t = -\theta(Z_t - \bar{Z})dt + \hat{\sigma}dB_t^0$$

- Value $v = v(t, a, z, g, Z)$

$$\begin{aligned}
 -\partial_t v + \rho v = & \underbrace{\max_c u(c) + \mathcal{L}[v|c](t, a, z)}_{\text{standard HJB continuation value}} \quad \underbrace{-\theta(Z - \bar{Z})v_Z + \frac{\hat{\sigma}^2}{2}v_{ZZ}}_{\text{direct effect of risk of } Z \text{ on } v} \\
 & + \underbrace{\iint_{z, a} \frac{\delta v(t, a, z, g, Z)}{\delta g}[\tilde{a}, \tilde{z}] \mathcal{L}^*[g|c^*](t, \tilde{a}, \tilde{z})}_{\text{change due to distribution dynamics}}
 \end{aligned}$$

Adding Aggregate Risk to the Master Equation

► Consider aggregate risk

- Agg. TFP follows a AR(1) – Ornstein-Uhlenbeck process

$$dZ_t = -\theta(Z_t - \bar{Z})dt + \hat{\sigma}dB_t^0$$

- Value $v = v(t, a, z, g, Z)$

$$\begin{aligned}
 -\partial_t v + \rho v = & \underbrace{\max_c u(c) + \mathcal{L}[v|c](t, a, z)}_{\text{standard HJB continuation value}} \quad \underbrace{-\theta(Z - \bar{Z})v_Z + \frac{\hat{\sigma}^2}{2}v_{ZZ}}_{\text{direct effect of risk of } Z \text{ on } v} \\
 & + \underbrace{\iint_{z, a} \frac{\delta v(t, a, z, g, Z)}{\delta g}[\tilde{a}, \tilde{z}] \mathcal{L}^*[g|c^*](t, \tilde{a}, \tilde{z})}_{\text{change due to distribution dynamics}}
 \end{aligned}$$

- Master equation doesn't change much: Why?

- No *direct effects* on individual states from agg. shocks: distribution g is not affected by dB_t^0
- If it were (portfolio problems, see later): becomes *second order* \Rightarrow Monster equation!

Outline

1. Krusell Smith Model
2. Primer on the Master equation
3. **Projection in Heterogeneous Agents Models**
4. Numerical results for KS98
5. Testing bounded-rationality assumption in KS98.
6. “Macrofinance”: portfolio choice and Second Order Master Equation

Projection assumption in Krusell-Smith (KS98)

- What do households need for decisions? *only* changes in prices (r, w)
 - ⇒ Do not care of distribution g *per se*, Only one moment, K , *the mean* of g to forecast prices

$$K = \iint_{a,z} a g(da, z) \qquad r = \alpha K^{\alpha-1} - \delta$$

- **Assumption:** Restricted belief (here as in KS98)

$$v(a, z, g, Z) = \bar{v}(a, z, K^h, Z) + \varepsilon$$

– Dimensionality reduction: $\mathbb{E}[\varepsilon \mid g \text{ s.t. } K = \iint a g(da, z)] = 0$ + information loss

- **Novelty:** Nice property in Lions-derivative for the Master equation

$$\text{with } K^h = \int_x h(x) g(dx) \qquad \frac{\delta v(x, g)}{\delta g}[\tilde{x}] \equiv \frac{d\bar{v}(x, K^h)}{dK^h} h(\tilde{x})$$

Projection in the Master equation

- With first-moment: $v(a, z, \mathbf{g}, \mathbf{Z}) \approx \bar{v}(a, z, \mathbf{K}, \mathbf{Z})$

$$\rho \bar{v} = \underbrace{\max_c u(c) + \mathcal{L}[\bar{v} | c]_{(a, z, \mathbf{K}, \mathbf{Z})}}_{\text{standard HJB continuation value}} \quad \underbrace{-\theta(\mathbf{Z} - \bar{\mathbf{Z}})\bar{v}_Z + \frac{\hat{\sigma}^2}{2}\bar{v}_{ZZ}}_{\text{direct effect of risk of } \mathbf{Z} \text{ on } \bar{v}}$$

$$+ \bar{v}_K \iint_{z, a} \underbrace{[r\tilde{a} + w\tilde{z} - c^*(\tilde{a}, \tilde{z}, \mathbf{K}, \mathbf{Z})]}_{\text{change in agents } (\tilde{a}, \tilde{z}) \text{ decisions}} g(d\tilde{a}, \tilde{z})$$

- Aggregation:

$$dK = \iint_{z, a} [r\tilde{a} + w\tilde{z} - c^*(\tilde{a}, \tilde{z}, \mathbf{K}, \mathbf{Z})] g(d\tilde{a}, \tilde{z})$$

$$dK = rK + w\bar{L} - \mathcal{C}(K, \mathbf{Z}|g) \quad \text{w/ agg conso: } \mathcal{C}(K, \mathbf{Z}|g) = \iint_{z, a} c^*(\tilde{a}, \tilde{z}, \mathbf{K}, \mathbf{Z}) g(d\tilde{a}, \tilde{z})$$

- Micro-foundation of the law of motion dK through aggregation
- But still dependence on g ! How to “get rid of it”?

Master Equation becomes a fusion of two familiar HJB equations

- With $v = v(a, z, \mathbf{g}, \mathbf{Z}) \approx \bar{v}(a, z, K, \mathbf{Z})$

$$\begin{aligned} \rho \bar{v} = \max_c u(c) + [wz + ra - c] \bar{v}_a + \lambda(\bar{v}(a, z', \cdot) - \bar{v}(a, z, \cdot)) \\ - \theta(Z - \bar{Z}) \bar{v}_Z + \frac{\hat{\sigma}^2}{2} \bar{v}_{ZZ} + \underbrace{[ZK^\alpha - \delta K - \mathcal{C}(K, \mathbf{Z} | \mathbf{g})]}_{=dK} \bar{v}_K \end{aligned}$$

Master Equation becomes a fusion of two familiar HJB equations

► With $v = v(a, z, \mathbf{g}, \mathbf{Z}) \approx \bar{v}(a, z, K, \mathbf{Z})$

$$\begin{aligned} \rho \bar{v} = \max_c u(c) + [wz + ra - c] \bar{v}_a + \lambda(\bar{v}(a, z', \cdot) - \bar{v}(a, z, \cdot)) \\ - \theta(Z - \bar{Z}) \bar{v}_Z + \frac{\hat{\sigma}^2}{2} \bar{v}_{ZZ} + \underbrace{[ZK^\alpha - \delta K - C(K, Z | \mathbf{g})]}_{=dK} \bar{v}_K \end{aligned}$$

• Looks like a fusion of two standard models

– RBC: $v = v(K, Z)$

$$\rho v = \max_c u(c) + [ZK^\alpha - \delta K - C] v_K - \theta(Z - \bar{Z}) v_Z + \frac{\hat{\sigma}^2}{2} v_{ZZ}$$

– Aiyagari: $v = v(a, z)$

$$\rho v = \max_c u(c) + [wz + ra - c] v_a + \lambda(v(a, z', \cdot) - v(a, z, \cdot))$$

Agents' decision and global dynamical system

- With $v = \bar{v}(a, z, K, Z)$, we get individual decisions:

$$c^*(\tilde{a}, \tilde{z}, K, Z) = \begin{cases} u'^{-1}(\bar{v}_a(\tilde{a}, \tilde{z}, K, Z)) \\ w(K, Z)z + r(K, Z)a \end{cases}$$

Agents' decision and global dynamical system

- With $v = \bar{v}(a, z, K, Z)$, we get individual decisions:

$$c^*(\tilde{a}, \tilde{z}, K, Z) = \begin{cases} u'^{-1}(\bar{v}_a(\tilde{a}, \tilde{z}, K, Z)) \\ w(K, Z)z + r(K, Z)a \end{cases}$$

- Thus, get a dynamical system for $x = (a, z, K, Z)$
- Kolmogorov Forward equation for $x \Rightarrow$ Gives a distribution $\tilde{g}(x)$

Dynamical system

"Master" KFE

Agents' decision and global dynamical system

- With $v = \bar{v}(a, z, K, Z)$, we get individual decisions:

$$c^*(\tilde{a}, \tilde{z}, K, Z) = \begin{cases} u'^{-1}(\bar{v}_a(\tilde{a}, \tilde{z}, K, Z)) \\ w(K, Z)z + r(K, Z)a \end{cases}$$

- Thus, get a dynamical system for $x = (a, z, K, Z)$
- Kolmogorov Forward equation for $x \Rightarrow$ Gives a distribution $\tilde{g}(x)$
- Easy to get \tilde{g} from Master Eq. operator $\mathcal{A}[\bar{v}|c^*] \Leftrightarrow \mathcal{A}^*[\tilde{g}|c^*]$

Dynamical system

"Master" KFE

Agents' decision and global dynamical system

- With $v = \bar{v}(a, z, K, Z)$, we get individual decisions:

$$c^*(\tilde{a}, \tilde{z}, K, Z) = \begin{cases} u'^{-1}(\bar{v}_a(\tilde{a}, \tilde{z}, K, Z)) \\ w(K, Z)z + r(K, Z)a \end{cases}$$

- Thus, get a dynamical system for $x = (a, z, K, Z)$
- Kolmogorov Forward equation for $x \Rightarrow$ Gives a distribution $\tilde{g}(x)$
- Easy to get \tilde{g} from Master Eq. operator $\mathcal{A}[\bar{v}|c^*] \Leftrightarrow \mathcal{A}^*[\tilde{g}|c^*]$

Dynamical system

"Master" KFE

- Consistency condition for rational-expectation equilibrium: \sim Bayes rules / Radon-Nykodim

$$g(da, z_j)|_{K, Z} = g(da, z_j|K, Z) = \frac{\tilde{g}(da, z_j, dK, dZ)}{\int_{\mathbb{X}} \tilde{g}(da, z_j, dK, dZ)}$$

- Using this g get $\mathcal{C}(K, Z|g) \Rightarrow$ all we needed!!

Summary and numerical methods

1. General Master equation for $v(a, z, g, Z)$
2. Master Equation with “projection”: $v = \bar{v}(a, z, K, Z)$
 - Start from guess $g(a, z)$, $\mathcal{C}(K, Z|g)$, and $dK^{(n)}$
 - Solve Master Equation (4 dims): finite difference methods
 - Get ind. decisions $c^*(a, z, K, Z)$ and operator $\mathcal{A}[\bar{v}]$
3. “Master”-Kolmogorov forward for $x = (a, z, K, Z)$
 - Solve for distribution \tilde{g} “for free” with $\mathcal{A}^*[\tilde{g}]$
 - Update g thanks to $\tilde{g} \rightarrow$ update $\mathcal{C}(K, Z|g)$
 - Update agg. dynamics until convergence:

$$dK^{(n+1)} = ZK^\alpha - \delta K - \mathcal{C}(K, Z|g)$$

Summary and numerical methods

1. General Master equation for $v(a, z, g, Z)$

2. Master Equation with “projection”: $v = \bar{v}(a, z, K, Z)$

- Start from guess $g(a,z)$, $\mathcal{C}(K, Z|g)$, and $dK^{(n)}$
- Solve Master Equation (4 dims): finite difference methods
- Get ind. decisions $c^*(a,z,K,Z)$ and operator $\mathcal{A}[\bar{v}]$

3. “Master”-Kolmogorov forward for $x = (a,z,K,Z)$

- Solve for distribution \tilde{g} “for free” with $\mathcal{A}^*[\tilde{g}]$
- Update g thanks to $\tilde{g} \rightarrow$ update $\mathcal{C}(K, Z|g)$
- Update agg. dynamics until convergence:

$$dK^{(n+1)} = ZK^\alpha - \delta K - \mathcal{C}(K, Z|g)$$

► General procedure

- No need for deep-learning/splines/polynomials: use “**standard**” finite difference methods
- **Aggregation** from micro-decisions \rightarrow potentially very non-linear!
- **Rational Expectation**: Does **not** rely on bounded-rationality assumption of KS98

Master-Equation with higher moments:

► Extension with second moments: $v(a, z, \mathbf{g}, \mathbf{Z}) \approx \bar{v}(a, z, K, K_2, KL, \mathbf{Z})$

- $K_2 = \mathbb{V}\text{ar}(a)$, $KL = \mathbb{C}\text{ov}(a, z)$.

$$\begin{aligned} \rho \bar{v} = \max_c u(c) + \mathcal{L}[\bar{v} | c] - \theta(Z - \bar{Z})\bar{v}_Z + \frac{\hat{\sigma}^2}{2}\bar{v}_{ZZ} + \underbrace{(ZK^\alpha - \delta K - \mathbb{E}^g[c^*])}_{=dK} \bar{v}_K \\ + \underbrace{\mathbb{C}\text{ov}^g(a, s^*)}_{dK_2} \bar{v}_{K_2} - (\lambda_1 + \lambda_2) \underbrace{\mathbb{C}\text{ov}^g(a, z)}_{dKL_\alpha - KL} \bar{v}_{KL} \end{aligned}$$

- System: $x = (a, z, K, K_2, KL, \mathbf{Z})$: $\bar{v}(x) \Rightarrow c^*(x)$ and $dK = \mathcal{S}(K, K_2, KL, \mathbf{Z} | g) dt$

Master-Equation with higher moments:

- Extension with second moments: $v(a, z, \mathbf{g}, \mathbf{Z}) \approx \bar{v}(a, z, K, K_2, KL, \mathbf{Z})$

- $K_2 = \mathbb{V}\text{ar}(a)$, $KL = \mathbb{C}\text{ov}(a, z)$.

$$\begin{aligned} \rho \bar{v} = \max_c u(c) + \mathcal{L}[\bar{v} | c] - \theta(Z - \bar{Z})\bar{v}_Z + \frac{\hat{\sigma}^2}{2}\bar{v}_{ZZ} + \underbrace{(ZK^\alpha - \delta K - \mathbb{E}^g[c^*])}_{=dK} \bar{v}_K \\ + \underbrace{\mathbb{C}\text{ov}^g(a, s^*)}_{dK_2} \bar{v}_{K_2} - (\lambda_1 + \lambda_2) \underbrace{\mathbb{C}\text{ov}^g(a, z)}_{dKL \propto -KL} \bar{v}_{KL} \end{aligned}$$

- System: $x = (a, z, K, K_2, KL, \mathbf{Z})$: $\bar{v}(x) \Rightarrow c^*(x)$ and $dK = \mathcal{S}(K, K_2, KL, \mathbf{Z}|g)dt$
- Theoretical insights: (numerics WIP)
 - With more moments (= information), one predict more accurately price dynamics (r, w)
 - Distribution over moments $\tilde{g}(x)$: self-regulating forces (e.g. KL)
 - If $\mathbb{C}\text{ov}^g(a, s^*) > 0 \Rightarrow K_2 \uparrow$, and if $\bar{v}_{K_2} < 0$, \rightarrow reinforces precautionary saving

Outline

1. Krusell Smith Model
2. Primer on the Master equation
3. Projection in Heterogeneous Agents Models
4. **Numerical results for KS98**
5. Testing bounded-rationality assumption in KS98.
6. “Macrofinance”: portfolio choice and Second Order Master Equation

Numerical results – Summary

► Risk:

- Idiosyncratic risk: Two-state Markov process for labor income shocks z
- Aggregate risk: Three-state Markov process for TFP Z , with $\sigma(Z) = 12\%$

► I compare three economies:

1. RBC (Brock Mirman) model

⇒ (v, g) value and distribution over aggregate capital and TFP (K, Z) .

2. Aiyagari model

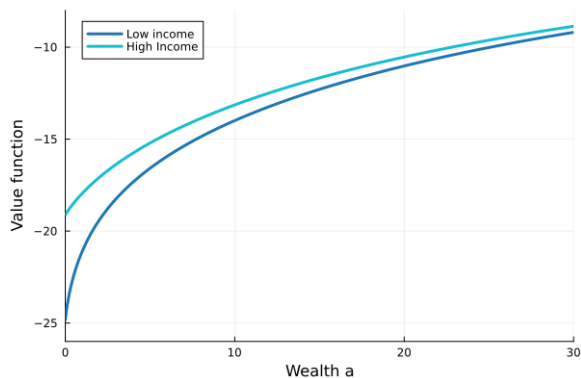
⇒ (v, g) individual heterogeneity on (a, z) , for constant TFP $Z = \bar{Z}$ and capital $K = \bar{K}$.

3. Krusell-Smith model

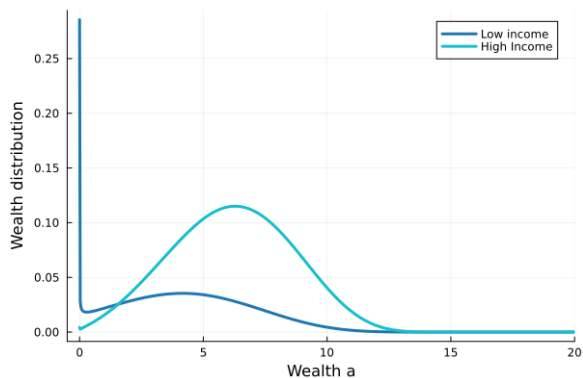
- Have both: (v, g) over (a, z, K, Z)
- Iterate over $dK = ZK^\alpha - \delta K - C(K, Z|g)$

Recap – Aiyagari model

Value function $v(a, z)$

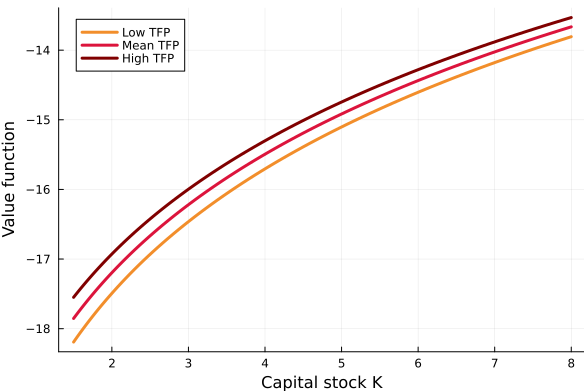


Distribution $g(a, z)$

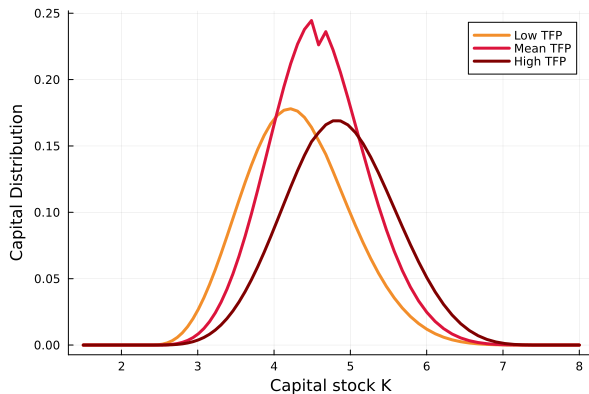


Recap – Brock-Mirman / RBC

Value function $v(K, Z)$

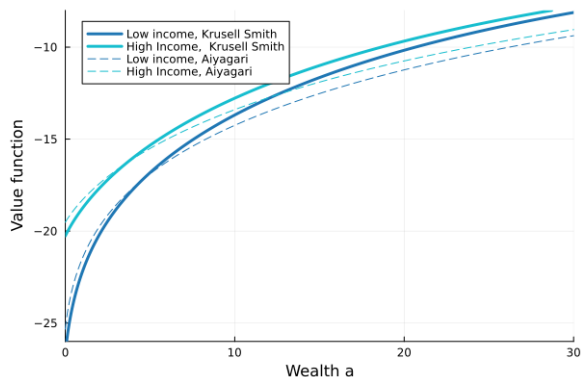


Distribution $g(K, Z)$

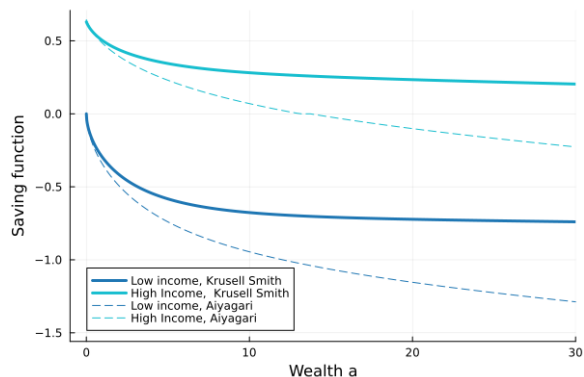


Krusell-Smith: individual decisions, vs. Aiyagari

Value function $v(a, z, \bar{K}, \bar{Z})$

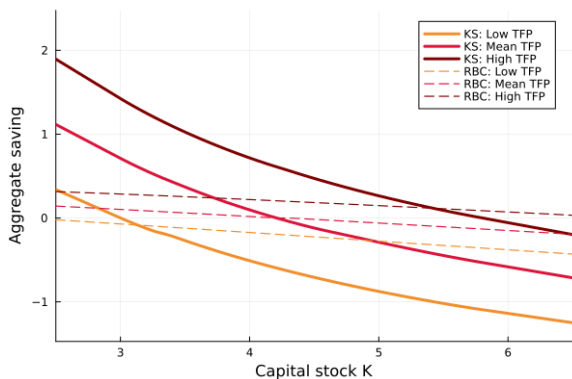


Saving $s(a, z, \bar{K}, \bar{Z}) = wz + ra - c^*$



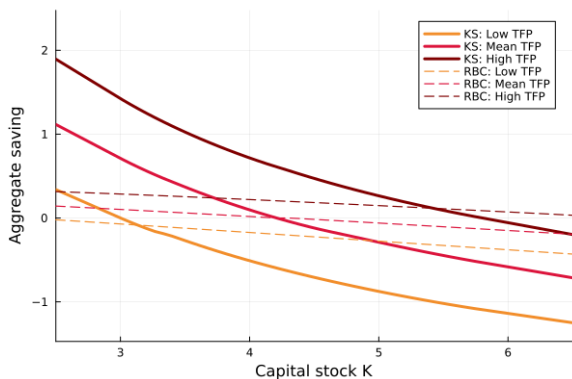
Krusell-Smith: aggregate dynamics decisions, vs. RBC

Aggregate dynamics $dK = ZK^\alpha - \delta K - C(K, Z|g)$

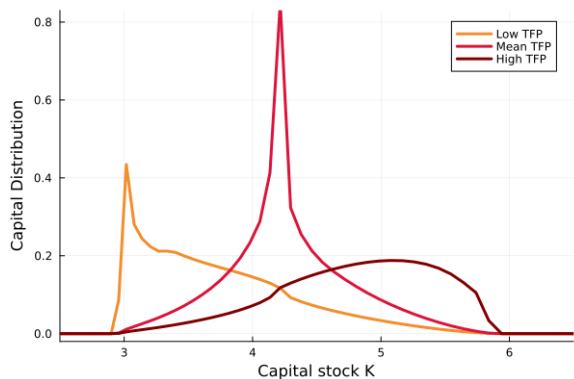


Krusell-Smith: aggregate dynamics decisions, vs. RBC

Aggregate dynamics $dK = ZK^\alpha - \delta K - C(K, Z|g)$

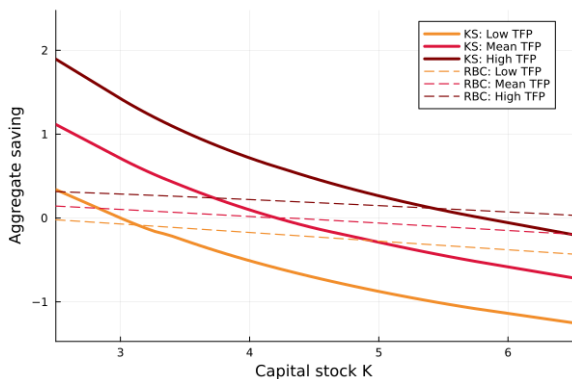


Distribution $\tilde{g}(K, Z)$

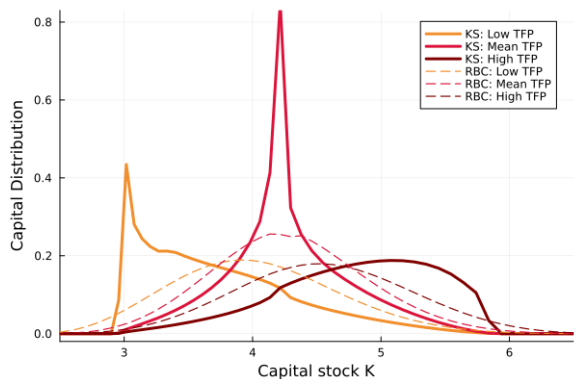


Krusell-Smith: aggregate dynamics decisions, vs. RBC

Aggregate dynamics $dK = ZK^\alpha - \delta K - C(K, Z|g)$



Distribution $\tilde{g}(K, Z)$



Effect of aggregate uncertainty and heterogeneity

1. Higher individual precautionary saving motives:

- Aggregate uncertainty affects the poor (reliant on labor income) more than the rich (hedging: $r = MPK \uparrow$ when $K \downarrow$)
- Implies much more savings when rich: flatter savings function compared to Aiyagari

Effect of aggregate uncertainty and heterogeneity

1. Higher individual precautionary saving motives:
 - Aggregate uncertainty affects the poor (reliant on labor income) more than the rich (hedging: $r = MPK \uparrow$ when $K \downarrow$)
 - Implies much more savings when rich: flatter savings function compared to Aiyagari
2. Aggregate dynamics are more “reactive”
 - Individual heterogeneity implies a steeper aggregate saving function
 - Rich household save a lot more when K is lower, boosting investment

Effect of aggregate uncertainty and heterogeneity

1. Higher individual precautionary saving motives:
 - Aggregate uncertainty affects the poor (reliant on labor income) more than the rich (hedging: $r = MPK \uparrow$ when $K \downarrow$)
 - Implies much more savings when rich: flatter savings function compared to Aiyagari
2. Aggregate dynamics are more “reactive”
 - Individual heterogeneity implies a steeper aggregate saving function
 - Rich household save a lot more when K is lower, boosting investment
3. Change the distribution of economic activity
 - Output is fluctuating more in Krusell-Smith: 15.5% volatility compared to 14% for RBC
 - However, business cycles are less skewed – smaller right tail, more symmetric – and have less kurtosis – thinner tails of capital/output.

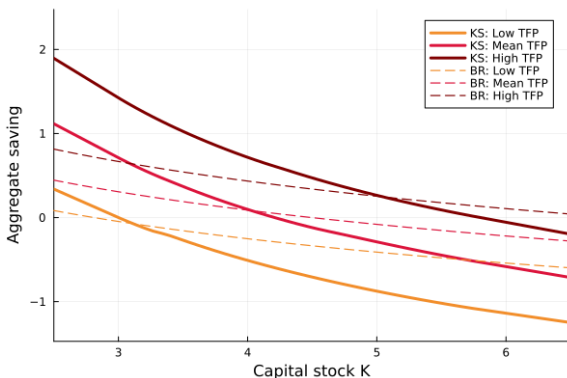
Outline

1. Krusell Smith Model
2. Primer on the Master equation
3. Projection in Heterogeneous Agents Models
4. Numerical results for KS98
5. **Testing bounded-rationality assumption in KS98**
6. “Macrofinance”: portfolio choice and Second Order Master Equation

Bounded-rationality in Krusell-Smith

- ▶ Agents do not forecast the “true” capital dynamics dK
 - Assume a log-linear rule: $dK = \beta^Z \log(K) dt$
 - What do we “lose” from this assumption?

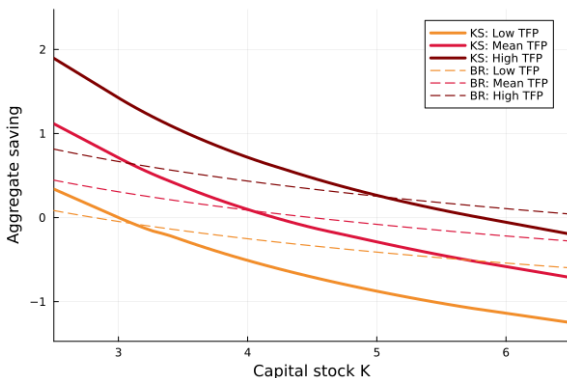
Aggregate dynamics $dK = \beta^Z \log(K) dt$



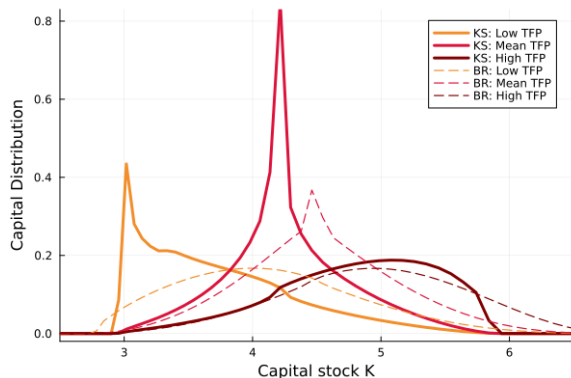
Bounded-rationality in Krusell-Smith

- Agents do not forecast the “true” capital dynamics dK
 - Assume a log-linear rule: $dK = \beta^Z \log(K) dt$
 - What do we “lose” from this assumption?

Aggregate dynamics $dK = \beta^Z \log(K) dt$



Distribution $\tilde{g}(K, Z)$



Outline

1. Krusell Smith Model
2. Primer on the Master equation
3. Projection in Heterogeneous Agents Models
4. Numerical results for KS98
5. Testing bounded-rationality assumption in KS98.
6. **“Macrofinance”**: portfolio choice and Second Order Master Equation

Master equation and Merton portfolio problem

- Agg. risk: both TFP (OU) $Z \sim \hat{\sigma} dB_t^0$ and *direct effect* on portfolio return (share α)

$$da = (ra + zw - c)dt + \alpha a (r^k - r)dt + \alpha a \bar{\sigma} dB_t^0$$

- The master equation now becomes **second order!** value $v = v(a, z, g, Z)$ changes a lot!

$$\rho v = \underbrace{\max_{c, \alpha} u(c) + \mathcal{L}[v|c, \alpha]_{(a, z, g, Z)}}_{\text{standard HJB continuation value}} \underbrace{-\theta(Z - \bar{Z})v_Z + \frac{\hat{\sigma}^2}{2} v_{ZZ}}_{\text{direct effect of risk of } Z \text{ on } v} + \underbrace{\iint_{z, a} \frac{dv(a, z, g, Z)}{dg} [\tilde{a}, \tilde{z}] \mathcal{L}^*[g|c^*]_{(\tilde{a}, \tilde{z})}}_{\text{deterministic evolution of the distribution}}$$

Master equation and Merton portfolio problem

- Agg. risk: both TFP (OU) $Z \sim \hat{\sigma} dB_t^0$ and *direct effect* on portfolio return (share α)

$$da = (ra + zw - c)dt + \alpha a (r^k - r)dt + \alpha a \bar{\sigma} dB_t^0$$

- The master equation now becomes **second order!** value $v = v(a, z, g, Z)$ changes a lot!

$$\begin{aligned} \rho v = & \underbrace{\max_{c, \alpha} u(c) + \mathcal{L}[v|c, \alpha]_{(a, z, g, Z)}}_{\text{standard HJB continuation value}} \underbrace{-\theta(Z - \bar{Z})v_Z + \frac{\hat{\sigma}^2}{2} v_{ZZ}}_{\text{direct effect of risk of } Z \text{ on } v} + \underbrace{\int \int_{z, a} \frac{dv(a, z, g, Z)}{dg} [\tilde{a}, \tilde{z}] \mathcal{L}^*[g|c^*]_{(\tilde{a}, \tilde{z})}}_{\text{deterministic evolution of the distribution}} \\ & + \underbrace{\frac{\bar{\sigma}^2}{2} \int \int_{z, a} \alpha(\tilde{a}, \tilde{z})^2 \tilde{a}^2 \frac{d}{d\tilde{a}} \left(\frac{dv}{dg} [\tilde{a}, \tilde{z}] \right) g(d\tilde{a}, \tilde{z})}_{\text{diffusion of the distribution due to risk}} + \underbrace{\bar{\sigma}^2 \alpha(a, z) \int \int_{z, a} \alpha(\tilde{a}, \tilde{z}) \tilde{a} \frac{d}{da} \frac{dv}{dg} [\tilde{a}, \tilde{z}] g(d\tilde{a}, \tilde{z})}_{\text{covariance of own state } a \text{ and distribution } \tilde{a}} \end{aligned}$$

Master equation and Merton portfolio problem

- Agg. risk: both TFP (OU) $Z \sim \hat{\sigma} dB_t^0$ and *direct effect* on portfolio return (share α)

$$da = (ra + zw - c)dt + \alpha a (r^k - r)dt + \alpha a \bar{\sigma} dB_t^0$$

- The master equation now becomes **second order!** value $v = v(a, z, g, Z)$ changes a lot!

$$\begin{aligned}
 \rho v = & \underbrace{\max_{c, \alpha} u(c) + \mathcal{L}[v|c, \alpha]_{(a, z, g, Z)}}_{\text{standard HJB continuation value}} \underbrace{-\theta(Z - \bar{Z})v_Z + \frac{\hat{\sigma}^2}{2} v_{ZZ}}_{\text{direct effect of risk of } Z \text{ on } v} + \underbrace{\iint_{z, a} \frac{dv(a, z, g, Z)}{dg} [\tilde{a}, \tilde{z}] \mathcal{L}^*[g|c^*]_{(\tilde{a}, \tilde{z})}}_{\text{deterministic evolution of the distribution}} \\
 & + \underbrace{\frac{\bar{\sigma}^2}{2} \iint_{z, a} \alpha(\tilde{a}, \tilde{z})^2 \tilde{a}^2 \frac{d}{d\tilde{a}} \left(\frac{dv}{dg} [\tilde{a}, \tilde{z}] \right) g(d\tilde{a}, \tilde{z})}_{\text{diffusion of the distribution due to risk}} + \underbrace{\bar{\sigma}^2 \alpha(a, z) \iint_{z, a} \alpha(\tilde{a}, \tilde{z}) \tilde{a} \frac{d}{da} \frac{dv}{dg} [\tilde{a}, \tilde{z}] g(d\tilde{a}, \tilde{z})}_{\text{covariance of own state } a \text{ and distribution } \tilde{a}} \\
 & + \underbrace{\bar{\sigma} \hat{\sigma} \iint_{z, a} \alpha(\tilde{a}, \tilde{z}) \frac{d}{dZ} \frac{dv}{dg} [\tilde{a}, \tilde{z}] g(d\tilde{a}, \tilde{z})}_{\text{covariance of agg. state } Z \text{ and distribution } \tilde{a}} + \underbrace{\frac{\bar{\sigma}^2}{2} \iint_{(z, a)^{\otimes 2}} \alpha(\tilde{a}, \tilde{z}) \alpha(\tilde{a}', \tilde{z}') \frac{d^2 \bar{v}}{dg^2} [\tilde{a}, \tilde{z}, \tilde{a}', \tilde{z}'] g(d\tilde{a}, \tilde{z}) g(d\tilde{a}', \tilde{z}')}_{\text{covariance of distribution } \tilde{a} \text{ and } \tilde{a}'}
 \end{aligned}$$

Master equation with projection for portfolio problem

- Agg. risk: both TFP (OU) $Z \sim \hat{\sigma} dB_t^0$ and *direct effect* on portfolio return (share α)
- With projection $v(a,z,g,Z) = \bar{v}(a,z,K,Z)$, master equation simplifies:

$$\begin{aligned}
 \rho \bar{v} = & \underbrace{\max_{c,\alpha} u(c) + \mathcal{L}[\bar{v}|c, \alpha]_{(a,z,K,Z)}}_{\text{standard HJB continuation value}} \quad \underbrace{-\theta(Z-\bar{Z})\bar{v}_Z + \frac{\hat{\sigma}^2}{2}\bar{v}_{ZZ}}_{\text{direct effect of risk of } Z \text{ on } v} \quad + \quad \underbrace{(ZK^\alpha - \delta K - \mathcal{C}(K, Z|g))\bar{v}_K}_{\text{deterministic evolution of the distribution}} \\
 & + \underbrace{\frac{\bar{\sigma}^2}{2} K \bar{v}_K \times 0}_{\text{diffusion of distribution due to risk} = 0} \quad + \quad \underbrace{\bar{\sigma}^2 \alpha(a,z,K,Z) a K \bar{v}_{Ka}}_{\text{covariance of own state } a \text{ and moment } K} \\
 & + \underbrace{\bar{\sigma} \hat{\sigma} K \bar{v}_{KZ}}_{\text{covar. of agg. state } Z \text{ and moment } K} \quad + \quad \underbrace{\frac{\bar{\sigma}^2}{2} K^2 \bar{v}_{KK}}_{\text{Var moment } K} \quad \text{with } K=A(K,Z|g)=\iint_{z,a} \alpha(\tilde{a}, \tilde{z}, \cdot) \tilde{a} g(d\tilde{a}, \tilde{z})
 \end{aligned}$$

Optimal decisions and portfolio

- ▶ Agg. risk: both TFP (OU) $Z \sim \hat{\sigma} dB_t^0$ and *direct effect* on portfolio return (share α)
 - With projection $v(a, z, g, Z) = \bar{v}(a, z, K, Z)$
 - Optimal decisions: consumption

$$c^*(a, z_j, K, Z) = \begin{cases} u'^{-1}(\bar{v}_a(a, z_j, K, Z)) & a > \underline{a} \\ z_j w(K, Z) + r a & \text{if } a \leq \underline{a} \end{cases}$$

- Optimal portfolio choice:

$$\alpha^*(a, z_j, K, Z) = \frac{r^k - r}{\gamma \bar{\sigma}^2} + \frac{\hat{\sigma}}{\bar{\sigma} \gamma} \frac{\bar{v}_{aZ}}{\bar{v}_a} + \frac{K \bar{v}_{aK}}{\gamma \bar{v}_a} \quad \text{if } a > \underline{a}$$

and $\alpha^*(a, z_j, K, Z) = 0$ if $a \leq \underline{a}$ with $\frac{1}{\gamma(x)} = -\frac{\bar{V}_a(a, z_j, K, Z)}{\bar{V}_{aa}(a, z_j, K, Z)a}$

Dynamic Oligopoly – Ericson-Pakes

- ▶ Firms i have cost state $x \in \mathbb{Z}$, investment a , market state $X = \{x_i\}_i$, and emp. distribution of firms $g = \sum_i \delta_{x_i}$, and aggregate cost Z
- ▶ Compete a la Bertrand, CES pref. σ , w/ profit $\pi(x, X) = \frac{Es_i}{\sigma - (\sigma - 1)s_i} - f$, market share $s_i = \left(\frac{p_i}{P}\right)^{1 - \sigma}$, static optimal pricing $p = \frac{\varepsilon}{\varepsilon - 1} xZ$, elasticity $\varepsilon = \sigma - (\sigma - 1)s_i$
- ▶ Master equation: $v = v(x, g, Z)$, with a discrete distribution / finite states

$$\rho v = \max_a \pi(x, X) - c(x)a + \mathcal{L}[v|a] + \int_x \frac{\delta V}{\delta g}[\tilde{x}] \mathcal{L}[g|a^*[\tilde{x}]$$

- ▶ With operator $\mathcal{L}[v|a] = \mu(x, a)v_x + \phi(x, a)(v(x_{i+1}, \cdot) - v(x_i, \cdot)) + \delta(v(x_{i-1}, \cdot) - v(x_i, \cdot))$

Conclusion

- ▶ I propose a new method for solving Heterogeneous Agent Models with aggregate risk
 - Leverage the Master equation approach and generalize Krusell-Smith insights with projections on finite sets of moments
- ▶ Next steps:
 - Additional example of information frictions models
 - Macro-finance model to the test + Replicate *Fernandez-Villaverde, Hurtado, Nuno (2023 Ecma)*
 - Oligopoly model a la Ericson Pakes (1995) with a distribution of firms within markets

Primer on the Lions derivative

- Derivative in the space of distribution: how the value $v(a, z, \mathbf{g})$ changes when the distribution of agents \mathbf{g} moves?

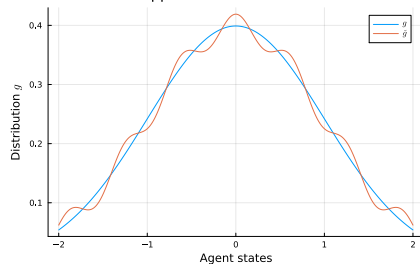
$$\begin{aligned}
 dv(a, z, \mathbf{g}) &\approx v(a, z, \tilde{\mathbf{g}}) - v(a, z, \mathbf{g}) \\
 &\approx \iint_{\tilde{a}, \tilde{z}} \underbrace{\frac{\partial v(a, z, \mathbf{g})}{\partial \mathbf{g}}[\tilde{a}, \tilde{z}]}_{=\text{Fréchet}} (\tilde{\mathbf{g}}(\tilde{a}, \tilde{z}) - \mathbf{g}(\tilde{a}, \tilde{z}))
 \end{aligned}$$

$$\approx \iint_{\tilde{a}, \tilde{z}} \underbrace{\frac{d}{d\tilde{a}} \frac{\partial v(a, z, \mathbf{g})}{\partial \mathbf{g}}[\tilde{a}, \tilde{z}]}_{=\text{Lions}} \underbrace{d\tilde{a}}_{=\text{change in decision}} \mathbf{g}(\tilde{a}, \tilde{z})$$

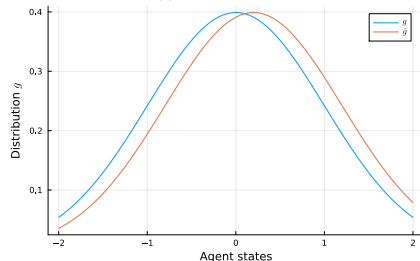
- $\frac{\partial v(a, z, \mathbf{g})}{\partial \mathbf{g}}[\tilde{x}]$ Fréchet Derivative, for a change of \mathbf{g} in \tilde{x}
- $\frac{dv(a, z, \mathbf{g})}{d\mathbf{g}}[\tilde{x}] = \frac{d}{dx} \frac{\partial v(a, z, \mathbf{g})}{\partial \mathbf{g}}[\tilde{x}]$ Lions Derivative, for a change of \tilde{x} , i.e. a *shift* in $\mathbf{g}(\tilde{x})$

[back](#)

Point of approximation: Fréchet derivative



Point of approximation: Lions derivative



Agents' decision and global dynamical system

► With $v = \bar{v}(a, z, K, Z)$, we get individual decisions:

$$c^*(\tilde{a}, \tilde{z}, K, Z) = \begin{cases} u'^{-1}(\bar{v}_a(\tilde{a}, \tilde{z}, K, Z)) \\ w(K, Z)z + r(K, Z)a \end{cases}$$

⇒ Dynamical system for $x = (a, z, K, Z)$

$$\begin{cases} da = [z \overbrace{(1-\alpha)ZK^\alpha}^{=w(K,Z)} + \overbrace{(\alpha ZK^{\alpha-1} - \delta)}^{=r(K,Z)} a - c^*(a, z, K, Z)] dt \\ dz = \gamma(z) dJ_t & \text{Markov, w/ intensity } \lambda(z) \\ dK = (ZK^\alpha - \delta K - \mathcal{C}(K, Z|g)) dt \\ dZ = -\theta(Z - \bar{Z}) dt + \hat{\sigma} dB_t^0 \end{cases}$$

• For a guess of $g(a, z)$ and $\mathcal{C}(K, Z|g)$ ⇒ complete characterization

⇒ Obtain a Kolmogorov Forward equation for system (a, z, K, Z) (!)

“Master-” Kolmogorov Forward for the global system

► For a guess of $g(a, z)$ and $\mathcal{C}(K, Z|g) = \iint_{a,z} c^*(a,z,K,Z) dg(a,z)$

- We can solve a “Master-KFE” for states $x = (a, z, K, Z) \in \mathbb{X}$ to find the distribution $\tilde{g}(x)$

$$0 = -\partial_a [s(x, \bar{v}_a) \tilde{g}(x)] + \sum_n \lambda(z^n) \tilde{g}(x^n) - \lambda(z) \tilde{g}(x) \\ - \partial_K [(ZK^\alpha - \delta K - \mathcal{C}(K, Z|g)) \tilde{g}(x)] - \partial_Z [-\theta(Z - \bar{Z}) \tilde{g}(x)] + \hat{\sigma} \partial_{ZZ}^2 \tilde{g}(x)$$

- Easy to get \tilde{g} from Master-HJB’s operator $\mathcal{A}[\bar{v}]$ with finite-difference methods

back