Heterogeneous Agents, Projection, and Aggregation
A Master Equation Approach

WORK IN PROGRESS

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Solving Heterogeneous Agents models with aggregate risk is hard

- Huge literature since the classic papers of Krusell-Smith (KS) and Den Haan
- Main difficulty:
  - With rational expectations, GE, and aggregate shocks, agents need to forecast the dynamics of prices and aggregate variables

  $\Rightarrow$ The distribution of agents enters the household/firm decision problem
  - It gives rise to a "Master equation", the value depends on an infinite-dimensional object g

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  - ⇒ The distribution of agents enters the household/firm decision problem
  - It gives rise to a "Master equation", the value depends on an infinite-dimensional object g
- Today's contribution:
  - Provide a method to solve this class of models with aggregate risk
  - Merge Krusell-Smith's original idea with modern treatments of the Master Equation
  - Allow to benchmark "non-rational expections" methods

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  - More recent methods bypassing this limitation – Second-order methods, *Bhandari, Bourany, Evans, Golosov (2023), Bayer, Luetticke, Weiss, Winkelmann (2025)*, Machine-Learning-based methods, *Fernandez-Villaverde, Hurtado, Nuno (2023), Huang (2023), Gu, Laurière, Merkel, Payne (2024)*, or others *Proehl (2019), Schaab (2021)* – may seem a little opaque or case-specific

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- But I borrow from the mathematics literature: Master equation
  - Mean Field Games w/ Common Noise, Master equation: *Cardaliaguet, Delarue, Lions, Lasry (2019)*
  - Also introduced in economics: *Schaab (2021), Bilal (2023), Gu, Laurière, Merkel, Payne (2024)*

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- Other applications (WIP)
    - Allows to benchmark "non-rational expections" methods
        - Moll (2025): Should we depart from RE in HA Models? *Sargent/Sims "wilderness of non-rational expectations"*
    - Extends to general macro-finance models/portfolio choice

Outline

1. Krusell Smith Model
2. Primer on the Master equation
3. Projection in Heterogeneous Agents Models
4. Numerical results for KS98
5. Testing bounded-rationality assumption in KS98.
6. “Macrofinance”: portfolio choice and Second Order Master Equation

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Krusell-Smith (1998) recap

- Consumption-saving model, c, a, with
  (i) idiosyncratic income risk z, (ii) incomplete market, (iii) credit constraints $a \geq \underline{a}$
  (iv) aggregate shock on aggregate TFP Z.
- Distribution of households $g(a, z)$ over wealth and income

Krusell-Smith (1998) recap

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  (iv) aggregate shock on aggregate TFP Z.
- Distribution of households $g(a, z)$ over wealth and income
- Firm side:

$$Y = ZK^{\alpha} \qquad \Rightarrow \qquad r = \alpha K^{\alpha-1} - \delta \qquad w = (1-\alpha)K^{\alpha}$$

- Household problem (KS98)

$$V(a, z, g, Z) = \max_{c, a'} \; u(c) + \beta \mathbb{E}^{z', Z'} \left[V(a', z', g', Z') \,\middle|\, z, Z \right]$$

$$s.t. \qquad c + a' = zw + (1+r)a$$

$$g' = H(g, Z, Z')$$

- Equilibrium

$$K = \int_{a,z} a \, dg(a, z) \qquad \forall \, Z$$

General idea and KS98 global solution

- Difficulty: Value function $V(a, z, g, Z)$ depends on the whole distribution g
- Need to forecast the evolution of $g \Rightarrow$ very difficult with aggregate risk
  - Need to follow distribution g_t on *every path* of $\{Z_t\}_t$
  - Brute force: computationally intensive, c.f. Achdou, Bourany (2018)
- Krusell-Smith solution
  1. Assume the Household only care about aggregate capital / First-moment $K = \int a\, dg(a, z)$
  2. Assume *Linear* forecasting rule for future capital

$$\log K' = a_1^Z \log K + a_2^Z$$

  - Choose parameters (a_1^Z, a_2^Z) to match the *realized / simulated* path (Monte Carlo) of $\{K_t\}_t$
- Proposal today:
  - remove assumption 2 $\Rightarrow$ bypass the linearity assumpt° (in that sense close to FVHN)
  - test robustness to 1 and 2, using methods based on the Master equation

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Primer on Mean Field Games and the Master Equation

- Aiyagari model rewrites as PDEs: MFG system
  - States dynamics: saving and labor income shocks

$$da_t = [z_t w_t + r_t\, a_t - c_t]dt \qquad\qquad z_j \sim \text{Markov jump process } \lambda_j$$

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1. Hamilton Jacobi Bellman Equation:

$$-\partial_t v(t,a,z) + \rho v(t,a,z) = \max_c\; u(c) + \mathcal{L}[\, v \,|\, c \,](t,a,z)$$

- Transport/Jump-Operator $\mathcal{L}$: *from agents' decision and shocks*

$$\mathcal{L}[\, v \,| c^\star](t,a,z_j) = \partial_a v(t,a,z_j) \underbrace{[z_j w + r\, a - c^\star]}_{\text{change in saving}} + \underbrace{\lambda_j \big(v(t,a,z_{-j}) - v(t,a,z_j)\big)}_{\text{change in labor income}}$$

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2. Kolmogorov Forward Equation:

$$\partial_t g(t,a,z) = \mathcal{L}^*[\, g \,|c^\star](t,a,z)$$

  - $\mathcal{L}^*$: *distribution dynamics comes from agents' decisions*

- Equilibrium:

$$\iint_{z,a\geq \underline{a}} a\, dg(t,a,z_j) = K \qquad\qquad r = \alpha K^{\alpha-1} - \delta$$

Primer on the Master Equation

- The master equation combines in *one equation* both HJB and KFE
  - Case without aggregate risk *c.f. Cardaliaguet, Delarue, Lions, Lasry (2019), Schaab (2021), Bilal (2023)*

$$-\partial_t v(t,a,z,g) + \rho v(t,a,z,g) = \overbrace{\max_c u(c) + \mathcal{L}[v\,|c^\star](t,a,z)}^{\text{standard HJB continuation value}} + \underbrace{\iint_{z,a} \frac{dv(t,a,z,g)}{dg}[(\tilde{a},\tilde{z})]\,\mathcal{L}^*[g\,|c^\star](t,\tilde{a},\tilde{z})\,dg(t,\tilde{a},\tilde{z})}_{\text{change in } v \text{ due to the distribution dynamics}}$$

  - First part: HJB, how states (a,z) change agents' value v
  - Novelty: depends on how the distribution g changes the value v
    - Notice the forecast from agents (a,z) about all the other agents $(\tilde{a},\tilde{z})$

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    - Notice the forecast from agents (a,z) about all the other agents $(\tilde{a},\tilde{z})$
    1. $\mathcal{L}^*[g\,|c^\star]$: How the agents' decision change the distribution g
    2. dv/dg: How the distribution changes the value
    - Needs to define derivatives in space of measures $\frac{dv(g)[\tilde{x}]}{dg}$: Lions' derivative *Lions vs. Fréchet*

Adding Aggregate Risk to the Master Equation

- Consider aggregate risk
  - Agg. TFP follows a AR(1) – Ornstein-Uhlenbeck process

$$dZ_t = -\theta(Z_t - \bar{Z})dt + \widehat{\sigma} dB_t^0$$

  - The master equation doesn't change much: value $v = v(t,a,z,g,Z)$

$$-\partial_t v + \rho v = \overbrace{\max_c \ u(c) + \mathcal{L}[\, v \,|c](t,a,z)}^{\text{standard HJB continuation value}} \quad \overbrace{-\theta(Z-\bar{Z})v_Z + \frac{\widehat{\sigma}^2}{2} v_{ZZ}}^{\text{direct effect of risk of } Z \text{ on } v}$$

$$+ \underbrace{\iint_{z,a} \frac{dv(t,a,z,g,Z)}{dg}[(\tilde{a},\tilde{z})]\, \mathcal{L}^*[\, g \,|c^\star](t,\tilde{a},\tilde{z})\ dg(t,\tilde{a},\tilde{z})}_{\text{change due to distribution dynamics}}$$

  - Why?
    - Aggregate shocks don't have *direct effects* on individual states!
    - As a result, distribution g is not affected/deformed directly by shocks dB_t^0
    - If it were, it would become *second order*: much more complicated ⇒ Monster equation!
    - More later (portfolio problems) if time permits

General (Second Order) Master Equation

- Include controlled drift, diffusion, jump on individual states
  + mean-field interaction on drift, diffusion, and jump on aggregate states
- Encompasses most macro-finance models (e.g. portfolio choice)

$$\mathcal{H}(x,m,\mathcal{X},V,D_xV,D_{xx}V) = \max_c \; \mathcal{L}(x,m,\mathcal{X},c) + b(x,m,\mathcal{X},c)\cdot D_xV + \mathrm{Tr}\big([\sigma\sigma' + \overline{\sigma}\overline{\sigma}'](x,m,\mathcal{X},c)\; D_{xx}V\big)$$
$$\textstyle\sum_{n=1}^{n_J^i} \lambda^n(x,m,\mathcal{X},c)\Big(V^n(x+\gamma(x,m,\mathcal{X},c),x,m,\mathcal{X}) - V\Big)$$

$$\begin{aligned}
-\partial_t V + \rho V &= \mathcal{H}\big(x,m,\mathcal{X},V,D_xV,D_{xx}V,c^\star\big)\\
&+ \mu(m,\mathcal{X})\cdot D_{\mathcal{X}}V + \mathrm{Tr}\big(\widehat{\sigma}\widehat{\sigma}' D_{\mathcal{X}\mathcal{X}}V\big) + \sum_{n=1}^{n_J^0} \widehat{\lambda}^n(m,\mathcal{X})\Big(V\circ\widehat{\gamma}^n(m,\mathcal{X}) - V\Big)\\
&+ \int_{\mathbb{X}} D_mV(x,\cdot;y)\cdot D_p\mathcal{H}(y,\cdot)m(dy) + \int_{\mathbb{X}}\sum_{n=1}^{n_J^0} \lambda^n(y,\cdot)\Delta_m V(x,\cdot;y)\circ\gamma(y,\cdot)m(dy)\\
&+ \int_{\mathbb{X}} \mathrm{Tr}\Big[(\sigma\sigma'+\overline{\sigma}\overline{\sigma}')(y,\cdot)D_y\big(D_mV(x,m,\mathcal{X};y)\big)\Big](y,m,\mathcal{X})m(dy)\\
&+ 2\int_{\mathbb{X}} \mathrm{Tr}\big(\overline{\sigma}(x,\cdot)\overline{\sigma}(y,\cdot)' D_xD_mV(x,\cdot;y)\big)m(dy) + \int_{\mathbb{X}} \mathrm{Tr}\big(\overline{\sigma}(y,\cdot)\widehat{\sigma}(\mathcal{X}_t)' D_mD_{\mathcal{X}}V(x,m,\mathcal{X};y)\big)m(dy)\\
&+ \iint_{\mathbb{X}} \mathrm{Tr}\big(\overline{\sigma}(y,\cdot)\overline{\sigma}(y',\cdot)' D^2_{mm}V\big)(x,\cdot;y,y')m(dy)m(dy')
\end{aligned}$$

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Projection assumption in KS98

- Back to KS98. What do households need for decisions?
  - Require only changes in prices $(r, w) \Rightarrow$ do not care of the distribution g *per se*
  - Only need the change in one moment, ***the mean*** of g, to forecast future prices!

$$K = \iint_{a,z} a \, dg(a,z) \qquad\qquad r = \alpha K^{\alpha-1} - \delta$$

  - Assumption (KS98 as well as here)

$$v(a,z,g,Z) \equiv \bar{v}(a,z,K^h,Z)$$

  - Novelty: Nice property in Lions-derivative for the Master equation

$$\text{with } K^h = \int_x h(x) \, dg(x) \qquad\qquad \frac{d}{dg} v(x,g)[\tilde{x}] \equiv \frac{d}{dK^h} \bar{v}(x,K^h) \, h'(\tilde{x})$$

    - First moment: $h(x) = x$, and $\frac{dv(x,g)}{dg} \equiv \frac{d\bar{v}(x,K^h)}{dK^h}$

Projection in the Master equation

- Rewrite Master Equation with projection on first-moment: $v(a,z,\mathbf{g},Z) \equiv \bar{v}(a,z,K,Z)$

$$\rho\bar{v} = \overbrace{\max_c u(c) + \mathcal{L}[\,\bar{v}\,|\,c\,]_{(a,z,K,Z)}}^{\text{standard HJB continuation value}} \overbrace{-\theta(Z-\bar{Z})\bar{v}_Z + \frac{\widehat{\sigma}^2}{2}\bar{v}_{ZZ}}^{\text{direct effect of risk of } Z \text{ on } \bar{v}}$$

$$+ \bar{v}_K \iint_{z,a} \underbrace{\left[r\tilde{a} + w\tilde{z} - c^\star(\tilde{a},\tilde{z},K,Z)\right]}_{\text{change in agents } (\tilde{a},\tilde{z}) \text{ decisions}} dg_{(\tilde{a},\tilde{z})}$$

  - Still dependence on g! How to "get rid of it"? (next slides!)
- Aggregation:

$$dK = \iint_{z,a} \left[r\tilde{a} + w\tilde{z} - c^\star(\tilde{a},\tilde{z},K,Z)\right] dg_{(\tilde{a},\tilde{z})}$$

$$dK = rK + w\bar{L} - \mathcal{C}(K,Z|g)$$

with aggregate consumption $\mathcal{C}(K,Z|g) = \iint_{z,a} c^\star(\tilde{a},\tilde{z},K,Z)\, dg_{(\tilde{a},\tilde{z})}$

The Master Equation becomes a fusion of two familiar equations

- The Master Equation becomes a “standard” HJB, $v = v(a, z, \mathbf{g}, Z) \equiv \bar{v}(a, z, K, Z)$

$$\rho\,\bar{v} = \max_c u(c) + [wz + ra - c]\,\bar{v}_a + \lambda\big(\bar{v}(a,z',\cdot) - \bar{v}(a,z,\cdot)\big)$$

$$- \theta(Z - \bar{Z})\bar{v}_Z + \frac{\widehat{\sigma}^2}{2}\bar{v}_{ZZ} + \underbrace{\left[ZK^\alpha - \delta K - \mathcal{C}(K, Z|g)\right]}_{=dK}\,\bar{v}_K$$

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  - Only issue: $\mathcal{C}(K, Z|g)$ still depends on g (next slide!)
  - Looks exactly like the fusion of two standard models
    - RBC: $v = v(K, Z)$

$$\rho v = \max_C u(C) + [ZK^\alpha - \delta K - C]v_K - \theta(Z - \bar{Z})v_Z + \frac{\widehat{\sigma}^2}{2} v_{ZZ}$$

    - Aiyagari: $v = v(a, z)$

$$\rho v = \max_c u(c) + [wz + ra - c]v_a + \lambda(v(a,z',\cdot) - v(a,z,\cdot))$$

Agents' decision and global dynamical system

- With the Master equation and $v = \bar{v}(a, z, K, Z)$ we obtain individual decisions:

$$c^{\star}(\tilde{a}, \tilde{z}, K, Z) = u'^{-1}\left(\bar{v}_a(\tilde{a}, \tilde{z}, K, Z)\right)$$

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$\Rightarrow$ get dynamical system for $x = (a, z, K, Z)$

$$\begin{cases} da &= \Big[z\,\overbrace{(1-\alpha)Z\,K^\alpha}^{=w} + \overbrace{(\alpha Z\,K^{\alpha-1} - \delta)}^{=r}\,a - c^\star(a, z, K, Z)\Big]dt \\ dz &= \gamma(z)dJ_t \qquad \text{Markov, w/ intensity} \qquad \lambda(z) \\ dK &= \left(ZK^\alpha - \delta K - \mathcal{C}(K, Z|g)\right)dt \\ dZ &= -\theta(Z - \bar{Z})dt + \widehat{\sigma}dB_t^0 \end{cases}$$

- For guess of $g_{(a,z)}$ and $\mathcal{C}(K, Z|g) = \iint_{a,z} c^\star{}_{(a,z,K,Z)}dg_{(a,z)} \Rightarrow$ complete characterization

$\Rightarrow$ Can get a Kolmogorov forward equation for system (a, z, K, Z) (!)

"Master-" Kolmogorov Forward for the global system

- For a guess of $g(a,z)$ and $\mathcal{C}(K,Z|g) = \iint_{a,z} c^\star(a,z,K,Z)\, dg(a,z)$
  - We can solve a "Master-KFE" for states $x = (a,z,K,Z) \in \mathbb{X}$ to find the distribution $\widetilde{g}(x)$

$$0 = -\partial_a\Big[s(x,\bar{v}_a)\widetilde{g}(x)\Big] + \textstyle\sum_n \lambda(z^n)\widetilde{g}(x^n) - \lambda(z)\widetilde{g}(x)$$

$$-\partial_K\Big[\big(ZK^\alpha - \delta K - \mathcal{C}(K,Z|g)\big)\widetilde{g}(\widetilde{x})\Big] - \partial_Z[-\theta(Z-\bar{Z})\widetilde{g}(x)] + \widehat{\sigma}\partial^2_{ZZ}\widetilde{g}(x)$$

  - Easy to get $\widetilde{g}$ from Master-HJB's operator $\mathcal{A}[\bar{v}]$ with finite-difference methods
- Consistency condition for rational-expectation equilibrium:

$$g(a,z)\Big|_{K,Z} = \frac{\widetilde{g}(a,z,K,Z)}{\widetilde{g}(K,Z)} \qquad\qquad \widetilde{g}(K,Z) = \int_{\mathbb{X}} \delta_{\{\widetilde{K}=K,\widetilde{Z}=Z\}} d\widetilde{g}(a,z,\widetilde{K},\widetilde{Z})$$

  - Using this g, we can obtain $dK = \big(ZK^\alpha - \delta K - \mathcal{C}(K,Z|g)\big)dt$, $\Rightarrow$ all we needed!!

Summary and numerical methods

1. General Master equation: one equation for $v(a, z, g, Z)$
2. Master Equation with "projection": $v = \bar{v}(a, z, K, Z)$
   - Start from guess $g(a, z)$ and $\mathcal{C}(K, Z|g)$
   - Solve Master-HJB: standard finite difference methods
   - Get individual decisions $c^{\star}(a,z,K,Z)$ and operator $\mathcal{A}[\bar{v}]$ for $x = (a, z, K, Z)$
3. "Master"-Kolmogorov forward for $x = (a, z, K, Z)$
   - Solve for distribution $\widetilde{g}$ over all states (a, z, K, Z) for "free" with $\mathcal{A}^*[\widetilde{g}]$
   - Update g thanks to $\widetilde{g}$ and update $\mathcal{C}(K, Z|g)$
   - Obtain aggregate dynamics: potentially very non-linear!!

$$dK = ZK^{\alpha} - \delta K - \mathcal{C}(K, Z|g)$$

- General procedure
  - No need for deep-learning/splines/polynomials: use "standard" finite difference methods
  - RE: Does not rely on bounded-rationality assumption of KS98, or forecasting rule

Master-Equation with higher moments:

- HJB with 2nd-order moments: $v(a,z,g,Z) \equiv \bar{v}(a,z,K,K_2,L_2,KL,Z) = \bar{v}(a,z,K,K_2,Z)$
  - $K_2 = \mathbb{V}\text{ar}(a)$, $L_2 = \mathbb{V}\text{ar}(z)$, $KL = \mathbb{C}\text{ov}(a,z)$.

$$\rho\,\bar{v} = \max_c\ u(c) + (wz + ra - c)\,\bar{v}_a + \lambda(\bar{v}_{(a,z',\cdot)} - \bar{v}_{(a,z,\cdot)}) - \theta(Z - \bar{Z})\bar{v}_Z + \frac{\widehat{\sigma}^2}{2}\bar{v}_{ZZ}$$

$$+ \underbrace{\left(ZK^{\alpha} - \delta K - \mathbb{E}^g[c^\star]\right)}_{=dK} \bar{v}_K + \underbrace{\mathbb{C}\text{ov}^g(a,s^\star)}_{dK_2} \bar{v}_{K_2}$$

  - Similarly, solve dynamical system (a,z,K,K_2,Z), the “master” KFE and then plug g back into $\mathbb{E}^g[c^\star] = \iint c^\star dg$ and $\mathbb{C}\text{ov}^g(a,s^\star) = \iint (a-\bar{a})(ra + wz - c^\star)dg$

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$$+ \underbrace{\big(ZK^\alpha - \delta K - \mathbb{E}^g[c^\star]\big)}_{=dK}\,\bar{v}_K + \underbrace{\mathbb{C}\mathrm{ov}^g(a,s^\star)}_{dK_2}\,\bar{v}_{K_2}$$

  - Similarly, solve dynamical system (a,z,K,K_2,Z), the "master" KFE and then plug g back into $\mathbb{E}^g[c^\star] = \iint c^\star dg$ and $\mathbb{C}\mathrm{ov}^g(a,s^\star) = \iint (a-\bar{a})(ra + wz - c^\star)dg$
  - Theoretical insights:
  – In KS98, you don't need all the moments!
    - $\bar{v}_{L_2}$ and $\bar{v}_{KL}$ drops from HJB: No change in variance in labor income, $dL_2 =$ and $dKL = 0$
  – If $\bar{v}_{K_2}<0$ and $\mathbb{C}\mathrm{ov}^g(a,s^\star)>0$, it reinforces the precautionary saving motive and lower value

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6. “Macrofinance”: portfolio choice and Second Order Master Equation

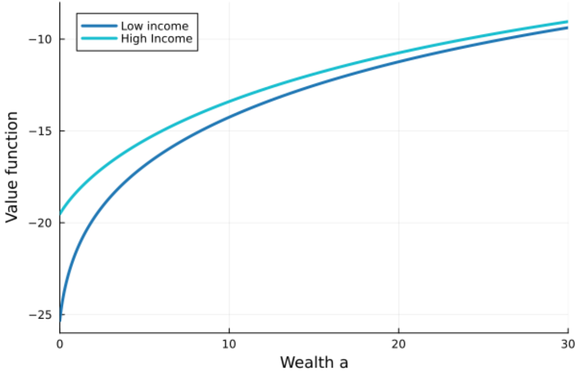
Numerical results – Summary

- Risk:
  - Idiosyncratic risk: Two-state Markov process for labor income shocks z
  - Aggregate risk: Three-state Markov process for TFP Z, with $\sigma(Z) = 12\%$
- I compare three economies:

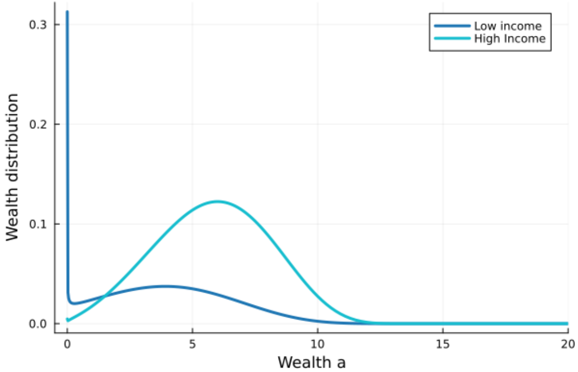
1. RBC (Brock Mirman) model
   $\Rightarrow$ (v, g) value and distribution over aggregate capital and TFP (K, Z).
2. Aiyagari model
   $\Rightarrow$ (v, g) individual heterogeneity on (a, z), for constant TFP $Z = \bar{Z}$ and capital $K = \bar{K}$.
3. Krusell-Smith model
   - Have both: (v, g) over (a, z, K, Z)
   - Iterate over $dK = ZK^{\alpha} - \delta K - \mathcal{C}(K, Z|g)$

Recap – Aiyagari model

Value function $v(a,z)$

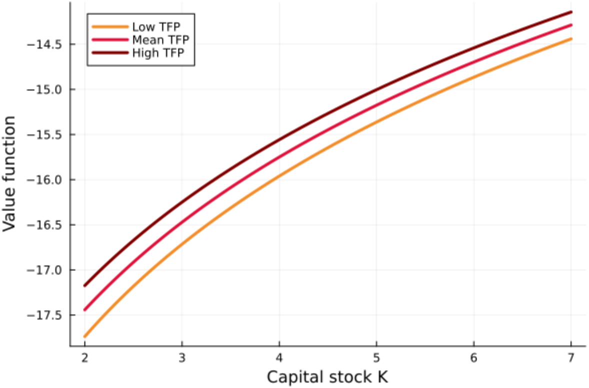


Distribution $g(a,z)$



Recap – Brock-Mirman / RBC

Value function $v(K,Z)$



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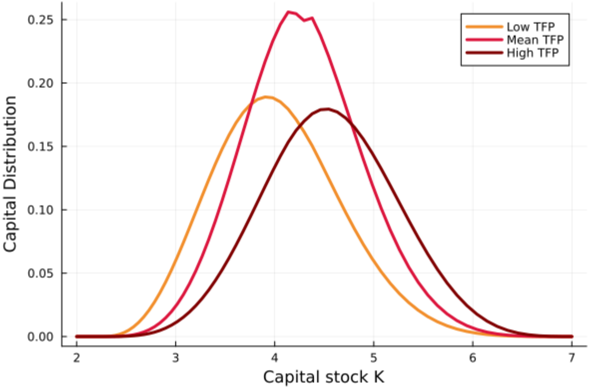
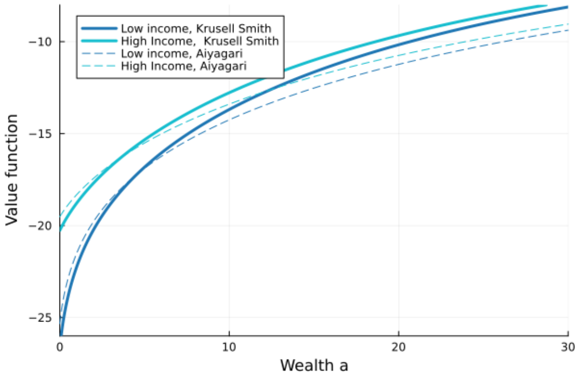
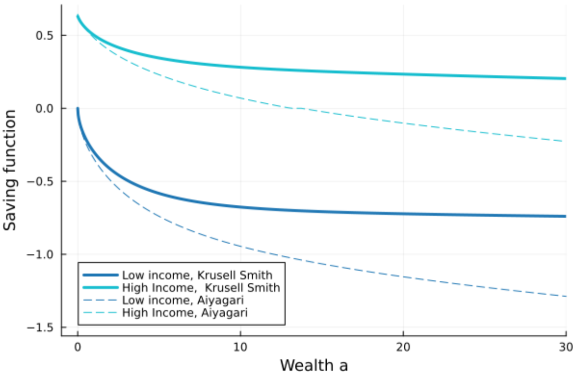


Krusell-Smith: individual decisions, vs. Aiyagari

Value function $v(a, z, \bar{K}, \bar{Z})$

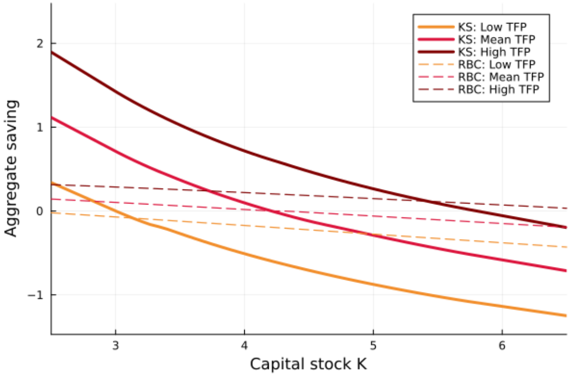


Saving $s(a, z, \bar{K}, \bar{Z}) = wz + ra - c^{\star}$



Krusell-Smith: aggregate dynamics decisions, vs. RBC

Aggregate dynamics $dK = ZK^{\alpha} - \delta K - \mathcal{C}(K, Z|g)$



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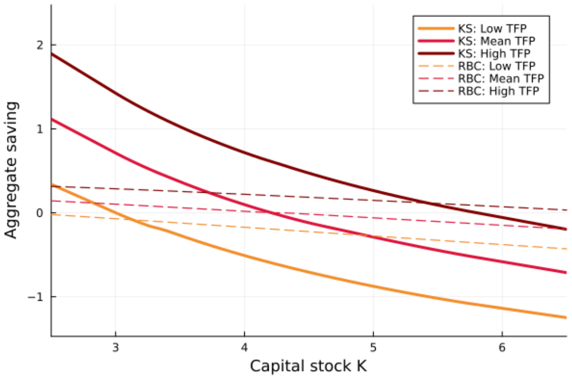
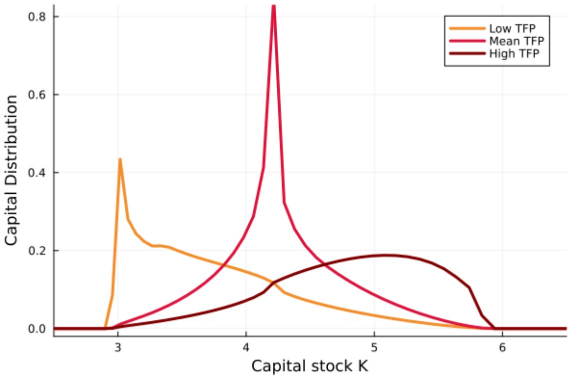


Distribution $\widetilde{g}(K, Z)$



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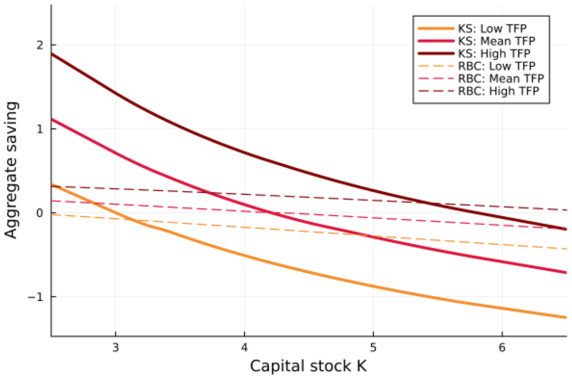

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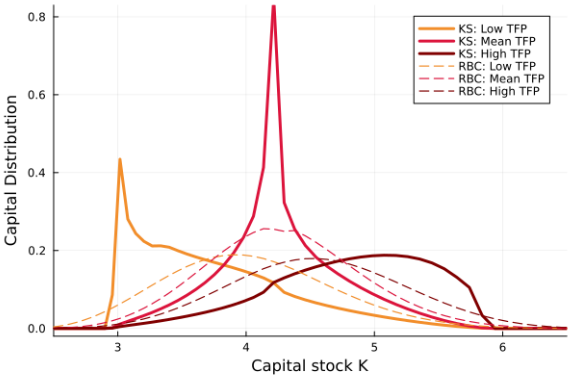

Effect of aggregate uncertainty and heterogeneity

1. Higher individual precautionary saving motives:
   - Aggregate uncertainty affects the poor (reliant on labor income) more than the rich (hedging: $r = MPK \uparrow$ when $K \downarrow$)
   - Implies much more savings when rich: flatter savings function compared to Aiyagari

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2. Aggregate dynamics are more "reactive"
   - Individual heterogeneity implies a steeper aggregate saving function
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3. Change the distribution of economic activity
   - Output is fluctuating more in Krusell-Smith: 15.5% volatility compared to 14% for RBC
   - However, business cycles are less skewed – smaller right tail, more symmetric – and have less kurtosis – thinner tails of capital/output.

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Bounded-rationality in Krusell-Smith

- Agents do not forecast the "true" capital dynamics dK
  - Assume a log-linear rule: $dK = \beta^Z \log(K)dt$
  - What do we "lose" from this assumption?

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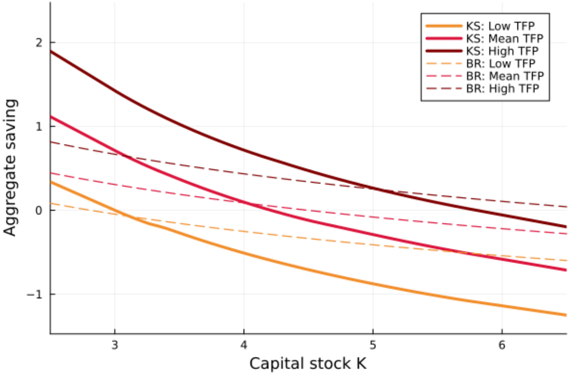

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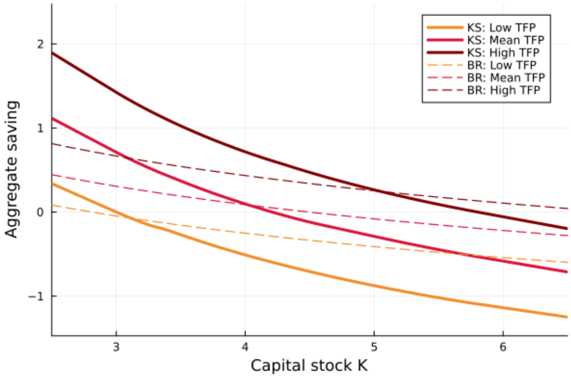

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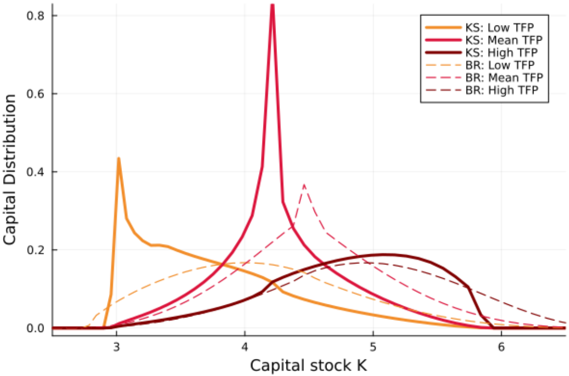

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Aggregate Risk and Master Equation for portfolio problem

- Agg. risk: both TFP (OU) $Z \sim \widehat{\sigma} dB_t^0$ and *direct effect* on portfolio return (share α)

$$da = (ra + zw - c)dt + \alpha\, a\,(R - r)dt + \alpha\, a\, \overline{\sigma} dB_t^0$$

  - The master equation now becomes ***second order***! value $v = v_{(a,z,\boldsymbol{g},Z)}$ changes a lot!

$$\rho v = \overbrace{\max_{c,\alpha} u(c) + \mathcal{L}[v|c,\alpha]_{(a,z,\boldsymbol{g},Z)}}^{\text{standard HJB continuation value}} \overbrace{-\theta(Z - \bar{Z})v_Z + \frac{\widehat{\sigma}^2}{2} v_{ZZ}}^{\text{direct effect of risk of } Z \text{ on } v} + \overbrace{\iint_{z,a} \frac{dv_{(a,z,\boldsymbol{g},Z)}}{dg}[\tilde{a},\tilde{z}]\, \mathcal{L}^*[\, g\, |c^\star]_{(\tilde{a},\tilde{z})} dg_{(\tilde{a},\tilde{z})}}^{\text{deterministic evolution of the distribution}}$$

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Conclusion

- I propose a new method for solving Heterogeneous Agent Models with aggregate risk
- Next steps:
    - Solving "more interesting" macro-finance models:
      Model with a meaningful distribution of portfolios, exposure, and impact of aggregate risk

Primer on the Lions derivative

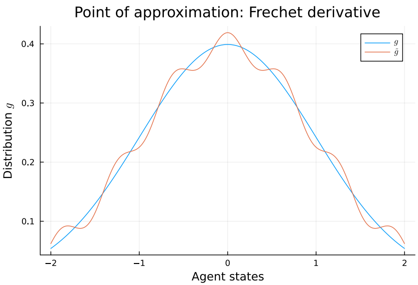
- Derivative in the space of distribution:
  how the value $v(a,z,\boldsymbol{g})$ changes when the distribution of agents $\boldsymbol{g}$ moves?

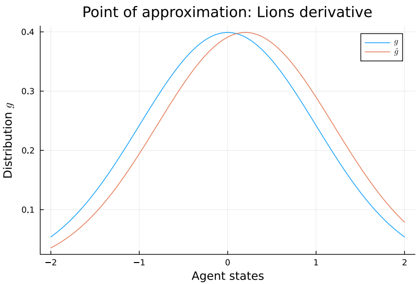
$$dv(a,z,\boldsymbol{g}) \approx v(a,z,\tilde{\boldsymbol{g}}) - v(a,z,\boldsymbol{g})$$

$$\approx \iint_{\tilde{a},\tilde{z}} \underbrace{\frac{\partial v(a,z,\boldsymbol{g})}{\partial g}[\tilde{a},\tilde{z}]}_{=\text{Fréchet}} \left(\tilde{g}(\tilde{a},\tilde{z}) - g(\tilde{a},\tilde{z})\right)$$

$$\approx \iint_{\tilde{a},\tilde{z}} \underbrace{\frac{d}{d\tilde{a}} \frac{\partial v(a,z,\boldsymbol{g})}{\partial g}[\tilde{a},\tilde{z}]}_{=\text{Lions}} \underbrace{d\tilde{a}}_{=\substack{\text{change in}\\ \text{decision}}} \quad g(\tilde{a},\tilde{z})$$

- $\frac{\partial v(a,z,\boldsymbol{g})}{\partial g}[\tilde{x}]$ Fréchet Derivative, for a change of g in $\tilde{x}$
- $\frac{dv(a,z,\boldsymbol{g})}{dg}[\tilde{x}] = \frac{d}{dx}\frac{\partial v(a,z,\boldsymbol{g})}{\partial g}[\tilde{x}]$ Lions Derivative, for a change of $\tilde{x}$, i.e. a *shift* in $g(\tilde{x})$





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