Lecture: Asset Pricing

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Asset pricing – Basics

Let's come back on the notations seen in Lecture 1:

Call p_t the price, $x_{t+1} = p_{t+1} + d_{t+1}$ the payoff (at time $t+1$), with d_t the dividend, and $R_t = \frac{x_t}{p_t}$ the Gross return of the asset, we have:

The fundamental equation of asset pricing:

$$p_t = \mathbb{E}_{\mathrm{t}}(m_{t+1}\, x_{t+1})$$

The expression of the risk-free rate:

$$R^f_{t+1} = 1 + r^f_{t+1} = \frac{1}{\mathbb{E}_{\mathrm{t}}(m_{t+1})}$$

The risk-premium of the asset i:

$$\mathbb{E}_{\mathrm{t}}(R^i_{t+1}) - R^f_{t+1} = \mathbb{E}_{\mathrm{t}}(r^i_{t+1}) - r^f_{t+1} = -R^f_{t+1}\, cov(m_{t+1}, R^i_{t+1})$$

Recall the "No-Arbitrage condition": $p^i_t = \frac{1}{R_t} = \frac{1}{1+r^i_{t+1}}$

Asset pricing – Basics - Arbitrage opportunities

- What is an *"Arbitrage opportunity"*? This arises when a new investment in a particular asset/portfolio V yields a non-negative return in all states-of-the-world, and (strictly) positive return in *some* states-of-the world.
  Formally: (in two-periods $t = 0, 1$), $V_0 = 0$ and $V_1 \geq 0$ (almost-surely), i.e. $\mathbb{P}(V_1 \geq 0) = 1$, and the proba of strict profit is positive, $\mathbb{P}(V_1 > 0) > 0$
- We say there is no arbitrage (NAO) when such portfolio/asset does not exists in the market, i.e. there is *no possibility to generate pure profit without risks.*
- Sounds like a realistic assumption... but arbitrage opportunities appears in case of... credit constraints! and other financial frictions (more on this in the course of "Financial economics")

Asset pricing – Basics - Risk-neutral probability

- Recall: a martingale is a sequence of random variable X_t which are integrable, adapted and s.t. $\mathbb{E}_t(X_{t+1}) = X_t$
- A probability $\tilde{\mathbb{P}}$ is a risk-neutral probability measure (or "martingale measure") if the asset$^{(*)}$ price S_t follows a martingale under this probability, or (said differently) if the expected value of future asset price is equal to present asset price. ***Formally:***

$$\mathbb{E}_t^{\tilde{\mathbb{P}}}(S_{t+1}) = S_t$$

  - How do you change the probability? Using the Girsanov theorem, we can introduce a new process Z_t such that

  $$\mathbb{E}_t^{\tilde{\mathbb{P}}}(X_{t+1}) = \mathbb{E}_t(Z_{t+1}X_{t+1})$$

  The process Z_t is called the pricing kernel, the stochastic discount factor or the Radon-Nikodym derivative.
- $(*)$ What asset? It is the asset you want to own in your portfolio and that is subjects to risky fluctuations. In Economics: the Arrow-Debreu Security. In Finance: the share/equity/derivative price.

Asset pricing – Basics - Pricing kernel

- What about economics?
- The risk-neutral probability measure is s.t.

$$\mathbb{E}_t(Z_{t+1}\, S_{t+1}) = \mathbb{E}_t^{\tilde{\mathbb{P}}}(S_{t+1}) = S_t$$

In our course, the pricing kernel (or stochastic discount factor) is denoted m_{t+1}

- What about consumption-based asset pricing? We know from the standard household optimization problem (Euler equation):

$$1 = \mathbb{E}_t(\beta \frac{u'(c_{t+1})}{u'(c_t)}\, R_t)$$

If we remember that we pay 1 for an asset yielding a gross interest rate $R_t = 1 + r_t$, it seems we have a good candidate for our pricing kernel!

$$m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

Asset pricing – Basics - 1st Fundamental theorem of pricing

- This results is not trivial, and comes from the ***First Fundamental theorem of asset pricing:***
  There exists such type of pricing kernel (or risk-neutral proba and that allows you to price any asset) *if and only if* the markets respects the No-Arbitrage-Opportunity (NAO) condition.

Asset pricing – Basics - Replicating portfolio

- An asset/derivative product A_t is said ***"replicable"*** if one can find a portfolio V_t that uses the assets on the markets (S_t), that is financed without external funds, and that can replicate the value of A at some time T, i.e. $V_T = A_T$ (almost surely, i.e. with proba one), starting from $V_0 = 0$.
- Such portfolio is called the ***replicating portfolio*** of the asset A or ***hedging*** portfolio. It is used to cover against the fluctuation of this particular asset A, using common assets S
  (in practice: such portfolio hedge against any asset prices fluctuations: derivative products, currencies/xrate risk, energy/oil price).
- We say the assets $S^{(1)}, S^{(2)} \ldots S^{(N)}$ "span" the market (all the assets A_t), i.e. we can find a linear combination of assets S that replicate any A.

Asset pricing – Basics - Complete markets

- We define "***complete markets***" a market where all assets A_t are replicable by a hedging portfolio.
- In an incomplete markets, there exists some assets that can not be replicated by hedging portfolio, or (equivalently) the market can not be "spanned" by the usual assets S: there are some risks uncovered in the market.
- In economics, by extension, we "define" the assets A as any random fluctuation that would affect the agent's payoff. Under such broad definition, unemployment shock ***is*** a shock that can not be replicated by a portfolio (e.g. by risk-free bonds).

- The ***second fundamental theorem of asset pricing*** says that markets are complete *if and only if* there exists a *unique* risk-neutral probability.
- In economics: in complete markets, it is impossible to have two different pricing kernels for the same asset (A). If not, one of the two would cover effectively the risks but not the other (who will suffer a loss (or a gain: arbitrage!)).

Asset pricing – Basics

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The risk-premium of the asset i:

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Recall the "No-Arbitrage condition": $p^i_t = \frac{1}{R_t} = \frac{1}{1+r^i_{t+1}}$

Asset pricing – Extension I

We can manipulate the formulas to provide alternative expressions:

The *beta-return* Representation:

$$\mathbb{E}_t(R^i_{t+1}) - R^f_{t+1} = \underbrace{-\frac{1}{\mathbb{E}_t(m_{t+1})}\, var(m_{t+1})}_{\lambda_m}\ \underbrace{\frac{cov(m_{t+1}, R^i_{t+1})}{var(m_{t+1})}}_{\beta_{i,m}}$$

$$\mathbb{E}_t(R^i_{t+1}) - R^f_{t+1} = \beta_{i,m}\, \lambda_m$$

where λ_m is the price of risk (in terms of consumption) and $\beta_{i,m}$ the quantity of risk for the asset i.

Asset pricing – Extension II

Calling $\sigma(z_t) = var(z_t)$, we rewrite:

The mean-variance representation:

$$\mathbb{E}_t(R^i_{t+1}) - R^f_{t+1} = -\rho_{m,R^i}\,\frac{\sigma(m_{t+1})}{\mathbb{E}_t(m_{t+1})}\,\sigma(R^i_{t+1})$$

$$= \rho_{m,R^i}\,\lambda_m\,\sigma(R^i_{t+1})$$

Because the correlation coefficient $\rho_{m,R^i} \in [-1, 1]$, we thus have bounds for the risk-premium:

$$\mathbb{E}_t(R^i_{t+1}) - R^f_{t+1} \leq \lambda_m\,\sigma(R^i_{t+1})$$

This is called the *Mean-Variance frontier*: "how much mean return can you get for a given level of variance?"

Asset pricing – Extension III

The Sharpe ratio:

$$\left| \frac{\mathbb{E}_t(R^i_{t+1}) - R^f_{t+1}}{\sigma(R^i_{t+1})} \right| \leq \frac{\sigma(m_{t+1})}{\mathbb{E}_t(m_{t+1})} = \lambda_m$$

The Sharpe ratio is one of the measure of return of an investment adjusting for the risk, i.e. the variability of its return.

With first-order approximation, one can show that $\lambda_m = \gamma\,\sigma(\Delta c_{t+1})$
Therefore, we can rewrite:

$$\left| \frac{\mathbb{E}_t(R^i_{t+1}) - R^f_{t+1}}{\sigma(R^i_{t+1})} \right| \leq \gamma\,\sigma(\Delta c_{t+1})$$

$$\mathbb{E}_t(R^i_{t+1}) - R^f_{t+1} = \beta_{i,m}\,\gamma\,\sigma(\Delta c_{t+1})$$

These inequalities are exactly linked to the ”Equity Premium Puzzle”
$\Rightarrow$ ”Why is the Equity return so high compared to the low growth of consumption and low risk-aversion?

CAPM and Factor pricing models

Consumption-based models do not work well in practice. The idea of the CAPM and other factor pricing models is to link the stochastic discount factor m_{t+1} to a factor f_{t+1}, which could be a good proxy for aggregate marginal utility:

$$\beta \frac{u'(c_{t+1})}{u'(c_t)} \approx m_{t+1} = a + b' f_{t+1}$$

The idea of the CAPM is to use a factor that represents the "wealth portfolio return". Assume that the representative agent hold all assets on the market and call the return on this asset as r_t^m: it should directly affects consumption.

CAPM

With the CRRA utility function, we can approximate:
$\beta \frac{u'(c_{t+1})}{u'(c_t)} = \beta(\frac{c_{t+1}}{c_t})^{-\sigma} = \beta(1 + \frac{\Delta c_{t+1}}{c_t})^{-\sigma} \approx \beta(1 - \sigma \frac{\Delta c_{t+1}}{c_t})$

Therefore we can rewrite the regression above as:

$$\frac{\Delta c_{t+1}}{c_t} = \alpha + \gamma \, r^m_{t+1} + \varepsilon_{t+1}$$

where α captures the trend in consumption and γ its comovement with the market rate. ε_{t+1} captures wealth shocks as a white noise and orthogonal to any asset returns r^m_{t+1}.

In practice, this is a broad-based stock portfolio such as the value- or equally-weighted NYSE, S&P500, etc.

This also implies:

$$cov\left(\frac{\Delta c_{t+1}}{c_t}, r^m_{t+1}\right) = \gamma^{-1} \, var\left(\frac{\Delta c_{t+1}}{c_t}\right) = \gamma \, var(r^m_{t+1})$$

Now for for *any* asset i, one can rewrite the risk-premium seen above as:

$$\mathbb{E}_{\mathrm{t}}\left(r^i_{t+1}\right) - r^f_{t+1} = \beta\sigma\left(1 + r_{t+1}\right) cov\left(\frac{\Delta c_{t+1}}{c_t}, r^i_{t+1}\right)$$

In particular, for the market portfolio

$$\mathbb{E}_{\mathrm{t}}\left(r^m_{t+1}\right) - r^f_{t+1} = \beta\sigma\left(1 + r^f_{t+1}\right) cov\left(\frac{\Delta c_{t+1}}{c_t}, r^m_{t+1}\right)$$

Dividing the two previous expressions and using $cov\left(\frac{\Delta c_{t+1}}{c_t}, r^m_{t+1}\right) = \gamma^{-1}\, var\left(\frac{\Delta c_{t+1}}{c_t}\right)$ on finds

$$\frac{\mathbb{E}_{\mathrm{t}}\left(r^i_{t+1}\right) - r^f_{t+1}}{\mathbb{E}_{\mathrm{t}}\left(r^m_{t+1}\right) - r^f_{t+1}} = \frac{cov\left(\frac{\Delta c_{t+1}}{c_t}, r^i_{t+1}\right)}{\gamma^{-1}\, var\left(\frac{\Delta c_{t+1}}{c_t}\right)}$$

Or...

The return of asset i rewrites:

$$\mathbb{E}_t\left(r^i_{t+1}\right) = r^f_{t+1} + \beta^i\left[E_t\left(r^m_{t+1}\right) - r^f_{t+1}\right]$$

$$\text{with} \quad \beta^i = \frac{cov\left(\frac{\Delta c_{t+1}}{c_t}, r^i_{t+1}\right)}{\gamma^{-1}\, var\left(\frac{\Delta c_{t+1}}{c_t}\right)} = \frac{cov\left(r^m_{t+1}, r^i_{t+1}\right)}{\gamma\, var\left(r^m_{t+1}\right)}$$

With r^f_{t+1} is the safe asset known in period t. Therefore, we end up with a beta formulation, which is much more convenient!
The gain is that we do not consider $\frac{\Delta c_{t+1}}{c_t}$ but r^m_{t+1} which is easily measured at a high frequency.

Note that this model holds with many alternative formulations:

- Quadratic utility: $u(c_t) = -1/2(c_t - c^*)$
- Exponential utility: $u(c_t) = -e^{-\alpha c_t}$ with α the CARA parameter
- Dynamic programming, with recursive value function
- Log-linearized models
- Continuous time

Factor pricing

Recall simple case

$$E_t\left(r^r_{t+1}\right) - r_{t+1} = -\left(1 + r_{t+1}\right) cov\left(m_{t+1}, r^r_{t+1}\right)$$

Much empirical finance work focuses on expected return-beta representations of *linear* factor pricing models, of the form

$$E\left(r^i\right) = \gamma + \beta_{i,a}\lambda_a + \beta_{i,b}\lambda_b + ..i = 1, .., N$$

β are regression coefficients on **factors,** considered as what is relevant to price assets (see below)

$$r^i_t = a_i + \beta_{i,a} f^a_t + \beta_{i,b} f^b_t + \cdots + \varepsilon^i_t, \text{ for } i = 1, .., N \;\; t = 1..T$$

- The CAPM is a special case, where the only factor is the market portfolio excess return

$$f_t = r^m_{t+1} - r_{t+1}$$

- The C-CAPM is a special case where the only factor is consumption growth

$$f_t = \frac{\Delta C_{t+1}}{C_t}$$

- The Three-Factors model, by Fama-French, consider three type of factors to determine the return of a protfolio:

$$r^i_t = r^f_t + \beta_3(r^m_t - r^f_t) + \beta_s SMB + \beta_v HML + \varepsilon^i_t, \text{ for } i = 1,..,N \; t = 1..T$$

 The begin of the equation is analogous to the standard CAPM, but we had two more factors: SMB, standing for "Small Minus Big", i.e. measuring the difference of return of stocks with small vs. big *market capitalization*, and HML, standing for the difference "High Minus Low" accounting for the value of *Price-to-book ratio*.

As we described, the factors at the right hand side are **proxies of marginal utility growth for the marginal investor**, in a situation where consumption is not matching the data and asset prices (because of Equity premium puzzle).

Relevant factors:

- Measure of the state of the economy consumption, GDP growth, investment, interest rates,
- variables to forecast : marginal utility growth (news): term premium, stock return, dividend price ratio,

(See the good discussion in Cochrane 2001, Chapter 9)